Asymptotic behaviour of solutions of kinetic equation

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Abstract

We consider a particle moving in a one-dimensional potential, which is at first time-homogeneous. The first part comes from an article from Nicolas Fournier and Camille Tardif [FT18]. We see the asymptotic behaviour of the position process: it behaves as a Brownian motion for $\beta \geq 5$, a stable process for $\beta \in [1, 5)$ and as an integrated symmetric Bessel process if $\beta \in (0, 1)$. In the second part, we study the time-inhomogeneous case. Starting from the velocity process studied by Yoann Offret in his thesis [Off12], for the attractive case and above the critical line: $2\beta > \alpha + 1$, we prove that the position process behaves asymptotically as a time-changed Brownian motion.
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Chapter 1

Introduction

In this paper, we consider a one-dimensional stochastic kinetic model driven by a Brownian motion. Let us denote by X_t the one-dimensional process describing the position of a particle at time $t \geq 0$, having the speed V_t:

$$X_t = X_0 + \int_0^t V_s \, \mathrm{d}s.$$

The velocity process (V_t) is supposed to be a Brownian process in a potential $U(t, v)$:

$$\mathrm{d}V_t = \mathrm{d}B_t - \frac{1}{2}\partial_v U(t, V_t)\, \mathrm{d}t.$$

In the first part, the potentiel U is supposed to be independent of time and satisfying

$$\partial_v U(v) = -\beta \frac{\vartheta'}{\vartheta},$$

where $\beta > 0$ and $\vartheta : \mathbb{R} \to (0, +\infty)$ is an even function of class C^2 satisfying $\lim_{|v|\to\infty} |v| \, \vartheta(v) = 1$. All the results of this part come from [FT18].
In the second part, the potential is supposed to depend on time and to verify

$$U(t, v) = \begin{cases} \dfrac{-2\rho}{\alpha + 1} \dfrac{|v|^{\alpha+1}}{t^\beta}, & \text{if } \alpha \neq -1, \\ \dfrac{-2\rho \log(|v|)}{t^\beta}, & \text{if } \alpha = -1. \end{cases}$$

Here $\rho < 0$, $\alpha \geq 0$, and $\beta \in \mathbb{R}$ are such that $2\beta - (\alpha + 1) > 0$ in order to use results from [Off12]. What is the asymptotic behaviour of the position process $X_{t/\epsilon}$, as $\epsilon \to 0$? We give an answer in the second part.

Chapter 2

Asymptotic behaviour of solution of a time-homogeneous kinetic equation

2.1 Introduction and main results

Consider, for two random variables (X_0, V_0) and a Brownian motion $(B_t)_{t\geq 0}$ independent from (X_0, V_0), the stochastic kinetic model:

$$\begin{aligned} V_t &= V_0 + B_t - \frac{\beta}{2}\int_0^t F(V_s)\,\mathrm{d}s, \\ X_t &= X_0 + \int_0^t V_s\,\mathrm{d}s, \end{aligned} \tag{2.1}$$

where $\beta > 0$. Assume that the potential F is of the form

$$F = -\frac{\vartheta'}{\vartheta}, \text{ where } \vartheta : \mathbb{R} \to (0, +\infty) \text{ is an even function of class } C^2 \text{ satisfying } \lim_{|v|\to\infty} |v|\,\vartheta(v) = 1. \tag{2.2}$$

In particular F is C^1 and thus is locally Lipschitz. One can keep in mind the example $F : v \mapsto \dfrac{v}{1+v^2}$ which comes from $\vartheta : v \mapsto (1+v^2)^{-1/2}$. The system (2.1) could be seen as a model for a particle motion in a one-dimensional potential.

One can observe that, since the drift and the diffusion coefficient are locally Lipschitz, then (2.1) has a unique local strong solution and it is a Markov process (see [Theorem 3.1 p. 178 in WI81]).

Moreover,

Lemma 2.1.1. *If it exists, the invariant measure μ_β of the velocity process $(V_t)_{t\geq 0}$ is solution of $\frac{1}{2}\mu_\beta'' + \frac{\beta}{2}(F\mu_\beta)' = 0$ in the sense of distributions. The unique (up to constant) solution is*

$$\mu_\beta(\mathrm{d}v) = c_\beta(\vartheta(v))^\beta\,\mathrm{d}v, \tag{2.3}$$

with $c_\beta^{-1} = \begin{cases} \int_{\mathbb{R}}[\vartheta(v)]^\beta\,\mathrm{d}v < +\infty & \text{if } \beta > 1, \\ 1 & \text{if } \beta \in (0,1]. \end{cases}$

Proof. The infinitesimal generator of V is given by $Lf(x) = -\frac{\beta}{2}F(x)f'(x) + \frac{1}{2}f''(x)$. The measure μ_β is invariant if and only if for all functions $f \in D(L) \subset C^\infty(\mathbb{R})$, $\int Lf(x)\mu_\beta(\mathrm{d}x) = 0$ (see [Prop 4.5 p.293 in WI81]). It is equivalent to say that $\langle \frac{\beta}{2}(F\mu_\beta)' + \frac{1}{2}\mu_\beta'', f\rangle = 0$ for all $f \in D(L)$ i.e. $\frac{1}{2}\mu_\beta'' + \frac{\beta}{2}(F\mu_\beta)' = 0$ in the sense of distributions. □

Remark 2.1.1. μ_β is a probability measure for $\beta > 1$, by Riemann criterion, using (2.2).

For a family $((Z_t^\epsilon)_{t\geq 0})_{\epsilon\geq 0}$ of processes, the notation $(Z_t^\epsilon)_{t\geq 0} \overset{\text{f.d}}{\Longrightarrow} (Z_t^0)_{t\geq 0}$ is used if, for all finite subset $S \subset [0, +\infty)$, the vector $(Z_t^\epsilon)_{t\in S}$ converges in distribution towards $(Z_t^0)_{t\in S}$ as $\epsilon \to 0$, and the notation $(Z_t^\epsilon)_{t\geq 0} \overset{\mathcal{L}}{\Longrightarrow} (Z_t^0)_{t\geq 0}$ is used if the convergence in distribution holds in the usual sense for continuous processes.

The main results of this part are the following:

Theorem 2.1.1. *Consider $\beta > 0$ and let $(V_t, X_t)_{t\geq 0}$ be a solution to (2.1). Then, as ϵ converges to 0,*

i) If $\beta > 5$, $(\sqrt{\epsilon} X_{t/\epsilon})_{t\geq 0} \overset{f.d}{\Longrightarrow} (\sigma_\beta \beta_t)_{t\geq 0}$.

ii) If $\beta = 5$, $\left(\sqrt{\frac{\epsilon}{\log \epsilon}} X_{t/\epsilon}\right)_{t\geq 0} \overset{f.d}{\Longrightarrow} (\sigma_5 \beta_t)_{t\geq 0}$.

iii) If $\beta \in (1, 5)$, $(\sqrt[\alpha]{\epsilon} X_{t/\epsilon})_{t\geq 0} \overset{f.d}{\Longrightarrow} (\sigma_\beta S_t^{(\alpha)})_{t\geq 0}$, where $\alpha = (\beta + 1)/3$.

iv) If $\beta = 1$, $(|\epsilon \log \epsilon|^{3/2} X_{t/\epsilon})_{t\geq 0} \overset{f.d}{\Longrightarrow} (\sigma_1 S_t^{(2/3)})_{t\geq 0}$.

v) If $\beta \in (0, 1)$, $(\sqrt{\epsilon} V_{t/\epsilon}, \epsilon^{3/2} X_{t/\epsilon}) \overset{\mathcal{L}}{\Longrightarrow} \left(U_t^{(1-\beta)}, \int_0^t U_s^{(1-\beta)}\, \mathrm{d}s\right)_{t\geq 0}$.

Here $(\beta_t)_{t\geq 0}$ is a Brownian motion, $(S_t^{(\alpha)})_{t\geq 0}$ is a symmetric stable process with index $\alpha \in (0, 2)$ such that $\mathbb{E}\left[\exp(iuS_t^{(\alpha)})\right] = \exp(-t\,|u|^\alpha)$ and $(U_t^{(\delta)})_{t\geq 0}$ is a symmetric Bessel process of dimension $\delta \in (0, 1)$. For each $\beta \geq 1$ the constant $\sigma_\beta > 0$ is defined by

- $\sigma_\beta^2 = 8c_\beta \int_0^{+\infty} \vartheta^{-\beta}(v) \left[\int_v^{+\infty} u\vartheta^\beta(u)\, \mathrm{d}u\right]^2 \mathrm{d}v$, *if $\beta > 5$,*
- $\sigma_5^2 = \dfrac{4c_5}{27}$,
- $\sigma_\beta^\alpha = \dfrac{3^{1-2\alpha} 2^{\alpha-1} c_\beta \pi}{\Gamma(\alpha)^2 \sin(\pi\alpha/2)}$, *with $\alpha = (\beta + 1)/3$, if $\beta \in (1, 5)$,*
- $\sigma_1^{2/3} = \dfrac{2^{2/3} 3^{-5/6} \pi}{\Gamma(2/3)^2}$.

Then one deduces the

Corollary 2.1.2. *Using the same hypotheses and notations as in the previous theorem, if $\tilde{V}$ is a random variable with law μ_β independent of $X^{(\beta)}$, then, as ϵ converges to 0,*

i) If $\beta > 5$, for each $t \geq 0$, $(\sqrt{\epsilon} X_{t/\epsilon}, V_{t/\epsilon}) \overset{\mathcal{L}}{\Longrightarrow} (\sigma_\beta \beta_t, \tilde{V})$.

ii) If $\beta = 5$, for each $t \geq 0$, $\left(\sqrt{\frac{\epsilon}{\log \epsilon}} X_{t/\epsilon}, V_{t/\epsilon}\right) \overset{\mathcal{L}}{\Longrightarrow} (\sigma_5 \beta_t, \tilde{V})$.

iii) If $\beta \in (1, 5)$, for each $t \geq 0$, $(\sqrt[\alpha]{\epsilon} X_{t/\epsilon}, V_{t/\epsilon}) \overset{\mathcal{L}}{\Longrightarrow} (\sigma_\beta S_t^{(\alpha)}, \tilde{V})$, where $\alpha = (\beta + 1)/3$.

2.2 Starting point

Introduce first some functions defined on $\mathbb{R}$:

- $h : v \mapsto (\beta + 1) \int_0^v \frac{1}{\vartheta(u)^\beta}\, \mathrm{d}u$. It is an odd, increasing, bijective function which solves $h'' = \beta F h'$. Integrating the equivalent given in (2.2), one gets $h(v) \underset{|v|\to\infty}{\sim} \mathrm{sgn}(v)\,|v|^{\beta+1}$ and $h^{-1}(v) \underset{|v|\to\infty}{\sim} \mathrm{sgn}(v)\,|v|^{1/(\beta+1)}$.

- $\sigma : z \mapsto h'\left(h^{-1}(z)\right)$. It is an even function, bounded from below by some $c > 0$. Besides, using the previous point, $\sigma(z) \underset{|z|\to\infty}{\sim} (\beta+1)\,|z|^{\beta/(\beta+1)}$.

- $\phi : z \mapsto \dfrac{h^{-1}(z)}{\sigma^2(z)}$. Since h^{-1} is an odd function, ϕ is, too. Using the two previous equivalents, one gets $\phi(z) \underset{|z|\to\infty}{\sim} \dfrac{\operatorname{sgn}(z)\,|z|^{(1-2\beta)/(\beta+1)}}{(\beta+1)^2}$.

- $g : v \mapsto 2\int_0^v \vartheta^{-\beta}(x) \int_x^{+\infty} u\vartheta^\beta(u)\,\mathrm{d}u\,\mathrm{d}x$. It is an odd function (using the fact that ϑ is even and that $\int_{\mathbb{R}} u\vartheta^\beta(u)\,\mathrm{d}u = 0$), satisfying the equation $g''(v) - \beta F(v)g'(v) = -2v$.

- $\psi : z \mapsto \dfrac{\left(g'\left(h^{-1}(z)\right)\right)^2}{\sigma^2(z)}$, when $\beta = 5$. It is an even and bounded function satisfying $\psi(z) \underset{|z|\to\infty}{\sim} \dfrac{1}{81\,|z|}$, thanks to the equivalent given in (2.2).

2.2.1 Reducing to the initial condition $(X_0, V_0) = (0, 0)$.

One can make the proof of Theorem 2.1.1 easier, by noting that it suffices to prove it when $X_0 = V_0 = 0$.

Lemma 2.2.1. *i) There exists $C > 0$ such that, if $V_0 = 0$, then for all $t \geq 0$, $\mathbb{E}[V_t^2 + |V_t|^{\beta+1}] \leq C(1+t)$.*

ii) Starting from any initial condition, the unique strong solution $(V_t)_{t\geq 0}$ is recurrent.

iii) If Theorem 2.1.1 is true when $X_0 = V_0 = 0$ a.s. and $\beta \geq 1$, then it is true for any initial condition.

iv) When $\beta \in (0,1)$, it suffices to prove that $(\sqrt{\epsilon}V_{t/\epsilon})_{t\geq 0} \overset{\mathcal{L}}{\Longrightarrow} (U_t^{(1-\beta)})_{t\geq 0}$ for $V_0 = 0$ in order to obtain Theorem 2.1.1 for any initial condition.

Proof. i) Set $\ell : v \mapsto 2\int_0^v \vartheta^{-\beta}(x)\int_0^x \vartheta^\beta(u)\,\mathrm{d}u\,\mathrm{d}x$. ϑ is even, then so is ℓ. Besides, ℓ satisfies

$$\ell''(v) - \beta F(v)\ell'(v) = 2.$$

Integrating the equivalent given in (2.2) (it suffices to study at $+\infty$ because ℓ is even), one gets that there exists a constant $c_\beta > 0$ such that:

- if $\beta > 1$, $\ell(v) \underset{|v|\to\infty}{\sim} c_\beta\,|v|^{\beta+1}$,
- if $\beta = 1$, $\ell(v) \underset{|v|\to\infty}{\sim} c_\beta v^2 \log|v|$,
- if $\beta \in (0,1)$, $\ell(v) \underset{|v|\to\infty}{\sim} c_\beta v^2$.

As a consequence, one can find a constant $c > 0$ such that, for any $\beta > 0$ and $v \in \mathbb{R}$, $v^2 + |v|^{\beta+1} \leq c(\ell(v)+1)$. Taking the expectation, one deduces $\mathbb{E}\left[V_t^2 + |V_t|^{\beta+1}\right] \leq c\left(\mathbb{E}[\ell(V_t)] + 1\right)$. Itô's formula and (2.1) yield

$$\ell(V_t) = \int_0^t \ell'(V_s)\,\mathrm{d}V_s + \frac{1}{2}\int_0^t \left(2 + \beta F(V_s)\ell'(V_s)\right)\mathrm{d}\langle V, V\rangle_s = \int_0^t \ell'(V_s)\,\mathrm{d}B_s + t.$$

Taking the expectation, one gets $\mathbb{E}\left[\ell(V_t)\right] = t$. This concludes the proof.

ii) The velocity process is a solution to a SDE with locally Lipschitz coefficients $b = -\beta F/2$ and $\sigma = 1$. But, using (2.2),

$$I := \int_{-\infty}^{0} \exp\left(-\int_0^x \beta \frac{\vartheta'(s)}{\vartheta(s)}\,\mathrm{d}s\right)\mathrm{d}x = \int_{-\infty}^{0} \exp\left(\beta \ln(\vartheta(0)/\vartheta(x))\right)\mathrm{d}x = \int_{-\infty}^{0} \left(\frac{\vartheta(0)}{\vartheta(x)}\right)^{\beta} \mathrm{d}x = +\infty$$

and, likewise,

$$J := \int_{0}^{+\infty} \exp\left(-\int_0^x \beta \frac{\vartheta'(s)}{\vartheta(s)}\,\mathrm{d}s\right)\mathrm{d}x = +\infty.$$

Thus, by [Proposition 5.22 p.345 in KS98], $(V_t)_{t\geq 0}$ is a recurrent process.

iii) **STEP 1: Find a solution to** (2.1) **starting from $(0,0)$.**
Assume $\beta \geq 1$ and suppose that Theorem 2.1.1 holds when the initial condition is $(0,0)$. Let $(V_t, X_t)_{t\geq 0}$ be the solution of (2.1) starting from some (V_0, X_0). Set $\tau = \inf\{t \geq 0, V_t = 0\}$. It is an almost surely finite stopping time by recurrence of V. Consider $(\hat{V}_t, \hat{X}_t)_{t\geq 0} := (V_{\tau+t} - V_\tau, X_{\tau+t} - X_\tau)_{t\geq 0}$. Since (V, X) is a Markov process, by strong Markov property, $\left(\hat{V}, \hat{X}\right)$ is independent from τ. Moreover $\hat{V}_\tau = 0$, $\hat{X}_\tau = 0$,

$$\hat{V}_t = V_{\tau+t} - V_\tau = B_{\tau+t} - B_\tau - \frac{\beta}{2}\int_0^t F(V_{\tau+s})\,\mathrm{d}s \overset{\mathcal{L}}{=} B_t - \frac{\beta}{2}\int_0^t F(\hat{V}_s)\,\mathrm{d}s,$$

since $V_\tau = 0$, and

$$\hat{X}_t = X_{\tau+t} - X_\tau = \int_\tau^{\tau+t} V_s\,\mathrm{d}s = \int_0^t \hat{V}_s\,\mathrm{d}s.$$

So, $(\hat{V}, \hat{X})$ is solution to (2.1) starting at $(0,0)$. Hence, one knows that $\left(v_\epsilon^{(\beta)} \hat{X}_{t/\epsilon}\right)_{t\geq 0} \overset{\text{f.d}}{\Longrightarrow} \left(X_t^{(\beta)}\right)_{t\geq 0}$, where the rate $v_\epsilon^{(\beta)}$ and the limit process $\left(X_t^{(\beta)}\right)_{t\geq 0}$ are given in the statement of Theorem 2.1.1.

STEP 2: For all $t \geq 0$, $v_\epsilon^{(\beta)} \left|X_{t/\epsilon} - \hat{X}_{t/\epsilon}\right| \xrightarrow{\mathbb{P}} 0$.
Fix $t \geq 0$. One has $\left|X_{t/\epsilon} - \hat{X}_{t/\epsilon}\right| \leq D^1 + D_t^{2,\epsilon}$, setting $D^1 = |X_0| + \int_0^{2\tau} |V_s|\,\mathrm{d}s$ and $D_t^{2,\epsilon} = \mathbb{1}_{\{t/\epsilon > \tau\}} \int_{t/\epsilon-\tau}^{t/\epsilon} \left|\hat{V}_s\right|\mathrm{d}s$. Indeed:

- if $t/\epsilon \leq \tau$,

$$\begin{aligned}\left|X_{t/\epsilon} - \hat{X}_{t/\epsilon}\right| &= \left|X_{t/\epsilon} - X_{\tau+t/\epsilon} + X_\tau\right| \leq \left|X_{t/\epsilon}\right| + \left|X_{\tau+t/\epsilon} - X_\tau\right| \\ &\leq |X_0| + \int_0^{t/\epsilon} |V_s|\,\mathrm{d}s + \int_\tau^{\tau+t/\epsilon} |V_s|\,\mathrm{d}s \\ &\leq |X_0| + \int_0^{\tau} |V_s|\,\mathrm{d}s + \int_\tau^{2\tau} |V_s|\,\mathrm{d}s = D^1 + D_t^{2,\epsilon}.\end{aligned}$$

- if $t/\epsilon > \tau$,

$$\begin{aligned}\left|X_{t/\epsilon} - \hat{X}_{t/\epsilon}\right| &= \left|X_\tau + X_{\tau+(t/\epsilon-\tau)} - X_\tau - \hat{X}_{t/\epsilon}\right| = \left|X_\tau + \hat{X}_{t/\epsilon-\tau} - \hat{X}_{t/\epsilon}\right| \\ &\leq |X_0| + \int_0^\tau |V_s|\,\mathrm{d}s + \left|\hat{X}_{t/\epsilon-\tau} - \hat{X}_{t/\epsilon}\right| \leq D^1 + \int_{t/\epsilon-\tau}^{t/\epsilon} \left|\hat{V}_s\right|\mathrm{d}s = D^1 + D_t^{2,\epsilon}.\end{aligned}$$

Since $\lim_{\epsilon\to 0} v_\epsilon^{(\beta)} = 0$, $v_\epsilon^{(\beta)} D^1 \underset{\epsilon\to 0}{\longrightarrow} 0$ a.s. and in probability, it remains to show that $v_\epsilon^{(\beta)} D_t^{2,\epsilon} \xrightarrow{\mathbb{P}} 0$, as $\epsilon \to 0$.

One can write,

$$\begin{aligned}\mathbb{E}\left[v_\epsilon^{(\beta)} D_t^{2,\epsilon}|\mathcal{F}_\tau\right] &= v_\epsilon^{(\beta)} \mathbb{1}_{\{t/\epsilon>\tau\}} \mathbb{E}\left[G(\tau,\hat{V})|\mathcal{F}_\tau\right], \text{ where } G:(s,v)\mapsto \int_{t/\epsilon-s}^{t/\epsilon} |v_u|\,\mathrm{d}u,\\ &= v_\epsilon^{(\beta)} \mathbb{1}_{\{t/\epsilon>\tau\}} \mathbb{E}\left[G(s,\hat{V})\right]_{|s=\tau}, \text{ since } \hat{V} \text{ is independent of } \tau,\\ &= v_\epsilon^{(\beta)} \mathbb{1}_{\{t/\epsilon>\tau\}} \int_{t/\epsilon-\tau}^{t/\epsilon} \mathbb{E}\left[\left|\hat{V}_u\right|\right] \mathrm{d}u \le v_\epsilon^{(\beta)} \mathbb{1}_{\{t/\epsilon>\tau\}} c \int_{t/\epsilon-\tau}^{t/\epsilon} (1+u)^{1/(\beta+1)}\,\mathrm{d}u\end{aligned}$$

because, Jensen inequality and the first point yield, for all $u \ge 0$,

$$\mathbb{E}\left[\left|\hat{V}_u\right|\right]^{\beta+1} \le \mathbb{E}\left[\left|\hat{V}_u\right|^{\beta+1}\right] \le \mathbb{E}\left[\hat{V}_u^2 + \left|\hat{V}_u\right|^{\beta+1}\right] \le c(1+u).$$

Hence,

$$\mathbb{E}\left[v_\epsilon^{(\beta)} D_t^{2,\epsilon}|\mathcal{F}_\tau\right] \le v_\epsilon^{(\beta)} \mathbb{1}_{\{t/\epsilon>\tau\}} c\tau(1+t/\epsilon)^{1/(\beta+1)} \le (\mathbb{1}_{\{t/\epsilon>\tau\}} c\tau)(\epsilon+t)^{1/(\beta+1)} v_\epsilon^{(\beta)} \epsilon^{-1/(\beta+1)}.$$

In any case, $\lim_{\epsilon\to 0} v_\epsilon^{(\beta)}\epsilon^{-1/(\beta+1)} = 0$, thus $\mathbb{E}\left[v_\epsilon^{(\beta)} D_t^{2,\epsilon}|\mathcal{F}_\tau\right] \underset{\epsilon\to 0}{\longrightarrow} 0$ almost surely.
Fix $\eta > 0$, by Markov's inequality,

$$\mathbb{P}\left(v_\epsilon^{(\beta)} D_t^{2,\epsilon} \ge \eta|\mathcal{F}_\tau\right) \le \frac{\mathbb{E}\left[v_\epsilon^{(\beta)} D_t^{2,\epsilon}|\mathcal{F}_\tau\right]}{\eta} \longrightarrow 0 \text{ almost surely.}$$

So, by the dominated convergence theorem, $\mathbb{P}\left(v_\epsilon^{(\beta)} D_t^{2,\epsilon} \ge \eta\right) \underset{\epsilon\to 0}{\longrightarrow} 0$ i.e. $v_\epsilon^{(\beta)} D_t^{2,\epsilon} \xrightarrow{\mathbb{P}} 0$. This concludes this step.

Step 3: Conclusion: $(v_\epsilon^{(\beta)} X_{t/\epsilon})_{t\ge 0} \overset{\text{f.d}}{\Longrightarrow} (X_t^{(\beta)})_{t\ge 0}$.
Fix $n \ge 0$ and $t_1, \cdots, t_n \ge 0$. By Slutsky lemma and the previous step, one has

$$v_\epsilon^{(\beta)} \sum_{i=1}^n \left|X_{t_i/\epsilon} - \hat{X}_{t_i/\epsilon}\right| \xrightarrow{\mathbb{P}} 0.$$

By step 1, $(v_\epsilon^{(\beta)}\hat{X}_{t_i/\epsilon})_{1\le i\le n} \overset{\mathcal{L}}{\Longrightarrow} (X_{t_i}^{(\beta)})_{1\le i\le n}$ so Lemma A.0.2 yields $(v_\epsilon^{(\beta)} X_{t_i/\epsilon})_{1\le i\le n} \overset{\mathcal{L}}{\Longrightarrow} (X_{t_i}^{(\beta)})_{1\le i\le n}$.

iv) **Step 1: The convergence of the velocity is sufficient.**
If, for any initial condition, one managed to prove that $(\sqrt{\epsilon}V_{t/\epsilon})_{t\ge 0} \overset{\mathcal{L}}{\Longrightarrow} (U_t^{(1-\beta)})_{t\ge 0}$, then $(\sqrt{\epsilon}V_{t/\epsilon}, \epsilon^{3/2}X_{t/\epsilon})_{t\ge 0} = G_\epsilon(\sqrt{\epsilon}V_{\cdot/\epsilon})$, where $G_\epsilon : v \mapsto \left(v_t, \epsilon^{3/2}X_0 + \int_0^t v_s\,\mathrm{d}s\right)_{t\ge 0}$ is converging uniformly to $G : v \mapsto \left(v_t, \int_0^t v_s\,\mathrm{d}s\right)_{t\ge 0}$, as $\epsilon \to 0$. So that, by Lemma A.0.3,

$$(\sqrt{\epsilon}V_{t/\epsilon}, \epsilon^{3/2}X_{t/\epsilon})_{t\ge 0} \overset{\mathcal{L}}{\Longrightarrow} (U_t^{(1-\beta)}, \int_0^t U_s^{(1-\beta)})_{t\ge 0}.$$

Assume now that one managed to show that for $V_0 = 0$, $(\sqrt{\epsilon}V_{t/\epsilon})_{t\ge 0} \overset{\mathcal{L}}{\Longrightarrow} (U_t^{(1-\beta)})_{t\ge 0}$. Consider $(V_t)_{t\ge 0}$ a solution to (2.1) starting at V_0. And, as in the preceding proof, introduce the stopping time τ and the process $\hat{V}$ which satisfies (2.1) and $\hat{V}_0 = 0$. Then, one gets $(\sqrt{\epsilon}\hat{V}_{t/\epsilon})_{t\ge 0} \overset{\mathcal{L}}{\Longrightarrow} (U_t^{(1-\beta)})_{t\ge 0}$.
Step 2: For all $T > 0$, $\delta_T^\epsilon := \sqrt{\epsilon}\sup_{[0,T]}\left|V_{t/\epsilon} - \hat{V}_{t/\epsilon}\right| \xrightarrow{\mathbb{P}} 0$.
Fix $T > 0$. Observe that $\sqrt{\epsilon}\sup_{[0,T]}\left|V_{t/\epsilon} - \hat{V}_{t/\epsilon}\right| = \sqrt{\epsilon}\sup_{[0,T]}\left|\hat{V}_{t/\epsilon-\tau} - \hat{V}_{t/\epsilon}\right|$. Fix $\eta > 0$, it suffices

to show that $\mathbb{P}\left(\sqrt{\epsilon}\sup_{[0,T]}\left|\hat{V}_{t/\epsilon-\tau}-\hat{V}_{t/\epsilon}\right|\geq\eta\right)\underset{\epsilon\to 0}{\longrightarrow}0$. Since $\left(\sqrt{\epsilon}\hat{V}_{t/\epsilon}\right)_{t\geq 0}$ converges in law in $C([0,+\infty))$, the family $\left\{\left(\sqrt{\epsilon}\hat{V}_{t/\epsilon}\right)_{t\geq 0},\ \epsilon>0\right\}$ is tight. Hence, by Proposition A.0.1,

$$\lim_{\delta\to 0}\sup_{\epsilon>0}\mathbb{P}(w_\delta(\sqrt{\epsilon}\hat{V}_{\cdot/\epsilon})\geq\eta)=0,$$

where $w_\delta(f):=\sup\{|f(t)-f(s);\ s,t\in[0,T],\ |t-s|\leq\delta|\}$, for every $f\in C([0,T])$. Fix $\gamma>0$. Let $\delta_0>0$ be chosen such that $\sup_{\epsilon>0}\mathbb{P}(w_{\delta_0}(\sqrt{\epsilon}\hat{V}_{\cdot/\epsilon})\geq\eta)\leq\gamma/2$. One can write,

$$\mathbb{P}\left(\sqrt{\epsilon}\sup_{[0,T]}\left|\hat{V}_{t/\epsilon-\tau}-\hat{V}_{t/\epsilon}\right|\geq\eta\right)\leq\mathbb{P}\left(\sqrt{\epsilon}\sup_{[0,T]}\left|\hat{V}_{t/\epsilon-\tau}-\hat{V}_{t/\epsilon}\right|\geq\eta,\epsilon\tau\leq\delta_0\right)+\mathbb{P}(\epsilon\tau>\delta_0).$$

Since τ is almost surely finite, $\epsilon\tau$ converges to 0 a.s. so in probability, consequently there exists ϵ_0 such that, for all $\epsilon\leq\epsilon_0$, $\mathbb{P}(\epsilon\tau>\delta_0)\leq\gamma/2$. On the other hand, on the event $\{\epsilon\tau\leq\delta_0\}$, for $t\in[0,T]$,

$$\sqrt{\epsilon}\left|\hat{V}_{t/\epsilon-\tau}-\hat{V}_{t/\epsilon}\right|\leq\sup_{\substack{t,s\in[0,T]\\|t-s|\leq\delta_0}}\sqrt{\epsilon}\left|\hat{V}_{s/\epsilon}-\hat{V}_{t/\epsilon}\right|=w_{\delta_0}(\sqrt{\epsilon}\hat{V}_{\cdot/\epsilon}).$$

So, $\sup_{[0,T]}\sqrt{\epsilon}\left|\hat{V}_{t/\epsilon-\tau}-\hat{V}_{t/\epsilon}\right|\mathbb{1}_{\{\epsilon\tau\leq\delta_0\}}\leq w_{\delta_0}(\sqrt{\epsilon}\hat{V}_{\cdot/\epsilon})$. Hence,

$$\begin{aligned}\mathbb{P}\left(\sqrt{\epsilon}\sup_{[0,T]}\left|\hat{V}_{t/\epsilon-\tau}-\hat{V}_{t/\epsilon}\right|\geq\eta,\epsilon\tau\leq\delta_0\right)&\leq\mathbb{P}\left(\sup_{[0,T]}\sqrt{\epsilon}\left|\hat{V}_{t/\epsilon-\tau}-\hat{V}_{t/\epsilon}\right|\mathbb{1}_{\{\epsilon\tau\leq\delta_0\}}\geq\eta\right)\\&\leq\mathbb{P}(w_{\delta_0}(\sqrt{\epsilon}\hat{V}_{\cdot/\epsilon})\geq\eta)\leq\gamma/2.\end{aligned}$$

This concludes this step.

STEP 3: $(\sqrt{\epsilon}V_{t/\epsilon})_{t\geq 0}\overset{\mathcal{L}}{\Longrightarrow}(U_t^{(1-\beta)})_{t\geq 0}$.

One knows that $(\sqrt{\epsilon}\hat{V}_{t/\epsilon})_{t\geq 0}\overset{\mathcal{L}}{\Longrightarrow}(U_t^{(1-\beta)})_{t\geq 0}$ and, for all $T>0$, $\sqrt{\epsilon}\sup_{[0,T]}\left|V_{t/\epsilon}-\hat{V}_{t/\epsilon}\right|\overset{\mathbb{P}}{\longrightarrow}0$.

Thus, $\mathrm{d}(\sqrt{\epsilon}V_{\cdot/\epsilon},\sqrt{\epsilon}\hat{V}_{\cdot/\epsilon})\overset{\mathbb{P}}{\longrightarrow}0$, where $d:f,g\in C([0,+\infty))\mapsto\sum_{n=0}^{+\infty}\frac{1}{2^n}\sup_{[0,n]}|f(t)-g(t)|$ is a metric on $C([0,+\infty))$. Indeed, fix $\eta>0$ and choose $N>0$ such that $\sum_{n=N+1}^{+\infty}1/2^n\leq\eta/2$, then,

$$\mathrm{d}(\sqrt{\epsilon}V_{\cdot/\epsilon},\sqrt{\epsilon}\hat{V}_{\cdot/\epsilon})\leq\eta/2+\sum_{n=0}^{N}\frac{1}{2^n}\sup_{[0,n]}\sqrt{\epsilon}\left|V_{t/\epsilon}-\hat{V}_{t/\epsilon}\right|.$$

It follows that

$$\mathbb{P}\left(\mathrm{d}(\sqrt{\epsilon}V_{\cdot/\epsilon},\sqrt{\epsilon}\hat{V}_{\cdot/\epsilon})>\eta\right)\leq\sum_{n=0}^{N}\mathbb{P}\left(\sup_{[0,n]}\sqrt{\epsilon}\left|V_{t/\epsilon}-\hat{V}_{t/\epsilon}\right|>\eta'\right)\underset{\epsilon\to 0}{\longrightarrow}0,$$

where $\eta'=\eta(2\sum_{n=N+1}^{+\infty}\frac{1}{2^n})^{-1}$. Lemma A.0.2 yields $(\sqrt{\epsilon}V_{t/\epsilon})_{t\geq 0}\overset{\mathcal{L}}{\Longrightarrow}(U_t^{(1-\beta)})_{t\geq 0}$.

□

2.2.2 Information on the velocity process with initial condition $(0,0)$.

In the sequel, $(W_t)_{t\geq 0}$ stands for a standard Brownian motion. Fix $\beta>0$, $\epsilon>0$, $a_\epsilon>0$ and define, for $t\geq 0$, $A_t^\epsilon=\frac{\epsilon}{a_\epsilon}\int_0^t\sigma\left(\frac{W_s}{a_\epsilon}\right)^{-2}\mathrm{d}s$. Since, for all $t\geq 0$, $A_t^{\epsilon\prime}=\frac{\epsilon}{a_\epsilon}\sigma\left(\frac{W_t}{a_\epsilon}\right)^{-2}$ is positive, then, $t\mapsto A_t^\epsilon$ is a continuous increasing function. Moreover, $A_0^\epsilon=0$ and by Lemma A.0.1, $A_\infty^\epsilon=+\infty$ almost surely. Thus, denoting by $(\tau_t^\epsilon)_{t\geq 0}$ its inverse, it is well defined, continuous increasing bijective from $\mathbb{R}^+$ to itself, thanks to the monotone bijection theorem. In order to prove that $(V_t)_{t\geq 0}$ is global, regular and recurrent, one needs the following lemma.

Lemma 2.2.2. *Set*

$$V_t^\epsilon := h^{-1}\left(\frac{W_{\tau_t^\epsilon}}{a_\epsilon}\right) \text{ and } X_t^\epsilon := H^\epsilon_{\tau_t^\epsilon}, \text{ where } H_t^\epsilon := \frac{1}{a_\epsilon^2}\int_0^t \phi\left(\frac{W_s}{a_\epsilon}\right) \mathrm{d}s.$$

If $(V_t, X_t)_{t\geq 0}$ *is the solution of* (2.1) *starting from* $(0,0)$ *then* $(V_{t/\epsilon}, X_{t/\epsilon})_{t\geq 0} \overset{\mathcal{L}}{=} (V_t^\epsilon, X_t^\epsilon)_{t\geq 0}$.

Remark 2.2.1. This lemma will be again useful for the proof of Theorem 2.1.1, by choosing the appropriate a_ϵ.

Proof. Set $Y_t^\epsilon := W_{\tau_t^\epsilon}$. There exists a Brownian motion $(B_t^\epsilon)_{t\geq 0}$ such that $(Y_t)_{t\geq 0}$ solves $Y_t^\epsilon = \frac{a_\epsilon}{\sqrt{\epsilon}}\int_0^t \sigma\left(\frac{Y_s^\epsilon}{a_\epsilon}\right) \mathrm{d}B_s^\epsilon$ (see [Proposition 1.13 p.373 in RY99] for details). By Itô's formula, one can write

$$V_t^\epsilon = V_0^\epsilon + \int_0^t (h^{-1})'\left(\frac{Y_s^\epsilon}{a_\epsilon}\right)\frac{\mathrm{d}Y_s^\epsilon}{a_\epsilon} + \frac{1}{2}\int_0^t (h^{-1})''\left(\frac{Y_s^\epsilon}{a_\epsilon}\right)\frac{\mathrm{d}\langle Y^\epsilon, Y^\epsilon\rangle_s}{a_\epsilon^2}.$$

But, $(h^{-1})'(y) = \dfrac{1}{\sigma(y)}$ and, using the equation satisfied by h,

$$(h^{-1})''(y) = \frac{-h''(h^{-1}(y))}{\sigma(y)} = \frac{-\beta F(h^{-1}(y))h'(h^{-1}(y))}{\sigma^3(y)} = \frac{-\beta F(h^{-1}(y))}{\sigma^2(y)}.$$

Thus,

$$V_t^\epsilon = \frac{1}{\sqrt{\epsilon}}B_t^\epsilon - \frac{\beta}{2\epsilon}\int_0^t F\left(h^{-1}\left(\frac{Y_s^\epsilon}{a_\epsilon}\right)\right)\mathrm{d}s = \frac{1}{\sqrt{\epsilon}}B_t^\epsilon - \frac{\beta}{2\epsilon}\int_0^t F(V_s^\epsilon)\,\mathrm{d}s.$$

On the other hand, using (2.1),

$$V_{t/\epsilon} = B_{t/\epsilon} - \frac{\beta}{2}\int_0^{t/\epsilon} F(V_s)\,\mathrm{d}s = \frac{1}{\sqrt{\epsilon}}(\sqrt{\epsilon}B_{t/\epsilon}) - \frac{\beta}{2\epsilon}\int_0^t F(V_{u/\epsilon})\,\mathrm{d}u.$$

Hence $(V_t^\epsilon)_{t\geq 0}$ and $(V_{t/\epsilon})_{t\geq 0}$ are solutions of two SDE driven by two Brownian processes $(B_t^\epsilon)_{t\geq 0}$ and $(\sqrt{\epsilon}B_{t/\epsilon})_{t\geq 0}$, so they have the same law, by [Theorem 3.5 ii) in RY99]. Besides, one gets

$$X_{t/\epsilon} = \int_0^{t/\epsilon} V_s\,\mathrm{d}s = \frac{1}{\epsilon}\int_0^t V_{s/\epsilon}\,\mathrm{d}s \overset{\mathcal{L}}{=} \frac{1}{\epsilon}\int_0^t V_s^\epsilon\,\mathrm{d}s,$$

and it follows that $\left(V_{t/\epsilon}, X_{t/\epsilon}\right)_{t\geq 0} \overset{\mathcal{L}}{=} \left(V_t^\epsilon, \frac{1}{\epsilon}\int_0^t V_s^\epsilon\,\mathrm{d}s\right)_{t\geq 0}$. To conclude, observe that

$$\begin{aligned}\frac{1}{\epsilon}\int_0^t V_s^\epsilon\,\mathrm{d}s &= \frac{1}{\epsilon}\int_0^t h^{-1}\left(\frac{W_{\tau_s^\epsilon}}{a_\epsilon}\right)\mathrm{d}s = a_\epsilon^{-2}\int_0^{\tau_t^\epsilon}\frac{h^{-1}(W_u/a_\epsilon)}{\sigma^2(W_u/a_\epsilon)}\,\mathrm{d}u\\ &= a_\epsilon^{-2}\int_0^{\tau_t^\epsilon}\phi(W_u/a_\epsilon)\,\mathrm{d}u = H_{\tau_t^\epsilon}.\end{aligned}$$

□

Definition 2.2.1. A process $(V_t)_{t\geq 0}$ is said to be *regular* if, for all $x, y \in \mathbb{R}$, $\mathbb{P}_x(T_y < \infty) > 0$, where $T_y = \inf\{t \geq 0, V_t = y\}$.

One is now able to obtain some information about the velocity process:

Lemma 2.2.3. *The solution* $(V_t)_{t\geq 0}$ *to* (2.1) *starting at* 0 *is global, regular and recurrent.*

Proof. Applying Lemma 2.2.2 with $a_\epsilon = \epsilon = 1$, one gets that $(V_t)_{t\geq 0}$ and $(h^{-1}(W_{\tau_t^1}))_{t\geq 0}$ have the same law, where $(\tau_t^1)_{t\geq 0}$ is a continuous time-change. Hence, (V_t) is defined for all times. By recurrence of the Brownian motion, since τ^1 and h are bijective, $(V_t)_{t\geq 0}$ is also recurrent. Moreover a Brownian motion is clearly regular, then so is the velocity process, by one-to-one correspondance. □

2.3 Proof of Theorem 2.1.1

2.3.1 Case $\beta > 5$

In this part, Theorem 2.1.1 is proved for the normal diffusive case $\beta > 5$. Assume $\beta > 5$, thanks to Lemma 2.2.1, one can assume $X_0 = V_0 = 0$. Since $\beta > 1$, (2.3) defines a probability measure and hence $(V_t)_{t\geq 0}$ is a positive recurrent process, having its invariant probability given by (2.3).

The function $g : v \mapsto 2\int_0^v \vartheta^{-\beta}(x)\int_x^{+\infty} u\vartheta^{\beta}(u)\,\mathrm{d}u\,\mathrm{d}x$, previously introduced, is an odd function satisfying $g''(v) - \beta F(v)g'(v) = -2v$. Itô's formula yields

$$g(V_t) = g(V_0)+\int_0^t g'(V_s)\,\mathrm{d}B_s-\int_0^t \frac{\beta}{2}g(V_s)F(V_s)\,\mathrm{d}s+\frac{1}{2}\int_0^t \beta F(V_s)g'(V_s)\,\mathrm{d}s-\int_0^t V_s\,\mathrm{d}s = \int_0^t g'(V_s)\,\mathrm{d}B_s-X_t,$$

because $X_0 = V_0 = 0$. It follows that $\sqrt{\epsilon}X_{t/\epsilon} = \sqrt{\epsilon}\int_0^{t/\epsilon} g'(V_s)\,\mathrm{d}B_s - \sqrt{\epsilon}g(V_{t/\epsilon})$.

Step 1: For all $t \geq 0$, $\sqrt{\epsilon}g(V_{t/\epsilon}) \xrightarrow{\mathbb{P}} 0$.
Thanks to [Lemma 23.17 p.466 in Kal02], V_t tends in distribution towards μ_β, as $t \to +\infty$. Fix $t \geq 0$, g is a continuous function, so $g(V_{t/\epsilon})$ converges weakly to $g(\tilde{V})$, as $\epsilon \to 0$, where $\tilde{V}$ is a μ_β-distributed random variable, hence, by Slutsky lemma, $\sqrt{\epsilon}g(V_{t/\epsilon}) \xrightarrow{\mathbb{P}} 0$.

Step 2: $(M_t^\epsilon)_{t\geq 0} \overset{\mathcal{L}}{\Longrightarrow} (\sigma_\beta\beta_t)_{t\geq 0}$, where $M_t^\epsilon := \sqrt{\epsilon}\int_0^{t/\epsilon} g'(V_s)\,\mathrm{d}B_s$.
By [Theorem 3.11 p. 473 in JS03], $(M_t^\epsilon)_{t\geq 0}$ being a continuous local martingale, it suffices to show that for all $t \geq 0$, $\langle M^\epsilon, M^\epsilon\rangle_t \xrightarrow{\mathbb{P}} \sigma_\beta^2 t$, as $\epsilon \to 0$. Fix $t \geq 0$, using Itô's isometry, $\langle M^\epsilon, M^\epsilon\rangle_t = \epsilon\int_0^{t/\epsilon} g'(V_s)^2\,\mathrm{d}s$. Besides, g'^2 is μ_β-integrable:

$$\int_{\mathbb{R}} g'(x)^2\mu_\beta(\mathrm{d}x) = 2\int_0^{+\infty} g'(x)^2\mu_\beta(\mathrm{d}x) = 8\int_0^{+\infty}\left[\vartheta^{-\beta}(x)\int_x^{+\infty} u\vartheta^{\beta}(u)\,\mathrm{d}u\right]^2\mu_\beta(\mathrm{d}x) = \sigma_\beta^2,$$

by definition of μ_β. Integrating the equivalent given in (2.2), σ_β^2 is finite, hence the ergodic theorem can be applied to find that

$$\epsilon\int_0^{t/\epsilon} g'(V_s)^2\,\mathrm{d}s = t\frac{\epsilon}{t}\int_0^{t/\epsilon} g'(V_s)^2\,\mathrm{d}s \underset{\epsilon\to 0}{\longrightarrow} t\int_{\mathbb{R}} g'^2\,\mathrm{d}\mu_\beta = \sigma_\beta^2 t.$$

Step 3: Conclusion.
Fix $n \geq 0$ and $t_1, \cdots, t_n \geq 0$. By Slutsky lemma $\sqrt{\epsilon}\sum_{i=1}^n \left|g(V_{t_i/\epsilon})\right| \xrightarrow{\mathbb{P}} 0$ and $(M_{t_i}^\epsilon)_{1\leq i\leq n} \overset{\mathcal{L}}{\Longrightarrow} (\sigma_\beta\beta_{t_i})_{1\leq i\leq n}$. Hence, by Lemma A.0.2, $(\sqrt{\epsilon}X_{t_i/\epsilon})_{1\leq i\leq n} \overset{\mathcal{L}}{\Longrightarrow} (\sigma_\beta\beta_{t_i})_{1\leq i\leq n}$.

This ends the proof of Theorem 2.1.1 i). □

Remark 2.3.1. For $\beta = 5$, the proof is the same, it remains to show that for all $t \geq 0$,
$\frac{\epsilon}{|\log\epsilon|}\int_0^{t/\epsilon} g'(V_s)^2\,\mathrm{d}s \xrightarrow{\mathbb{P}} \sigma_5^2 t$, as $\epsilon \to 0$.

2.3.2 Case $\beta \in (0, 1)$

In this part, Theorem 2.1.1 for $\beta \in (0, 1)$ is proved. Assume $\beta \in (0, 1)$, thanks to Lemma 2.2.1, it suffices to prove that $(\sqrt{\epsilon}V_{t/\epsilon})_{t\geq 0} \overset{\mathcal{L}}{\Longrightarrow} (U_t^{(1-\beta)})_{t\geq 0}$, when $V_0 = 0$.

Definition 2.3.1. Fix $\delta \in (0, 2)$. Set the time-change $\bar{A}_t := (2-\delta)^{-2}\int_0^t |W_s|^{-2(1-\delta)/(2-\delta)}\,\mathrm{d}s$ and its inverse $(\bar{\tau}_t)_{t\geq 0}$. Then $(\mathrm{sgn}(W_{\bar{\tau}_t})\,|W_{\bar{\tau}_t}|^{1/(2-\delta)})_{t\geq 0}$ is called *a symmetric Bessel process of dimension δ.*

Remark 2.3.2. Call $\alpha = \dfrac{2(1-\delta)}{(2-\delta)} < 1$. Then, for all $t \geq 0$,

$$\begin{aligned}\mathbb{E}\left[\int_0^t |W_s|^{-\alpha}\,\mathrm{d}s\right] &= \int_0^t \mathbb{E}\left[|W_s|^{-\alpha}\right]\mathrm{d}s = 2\int_0^t\int_0^{+\infty}\frac{x^{-\alpha}e^{-x^2/2s}}{\sqrt{2\pi s}}\,\mathrm{d}x\,\mathrm{d}s \\ &\leq 2\int_0^t\int_0^{+\infty}\frac{x^{-\alpha}e^{-x^2/2t}}{\sqrt{2\pi s}}\,\mathrm{d}x\,\mathrm{d}s = 2\sqrt{\frac{2t}{\pi}}\int_0^{+\infty}x^{-\alpha}e^{-x^2/2t}\,\mathrm{d}x < +\infty.\end{aligned}$$

Hence, $\mathbb{E}\left[\bar{A}_t\right] < +\infty$ almost surely. So, the map $t \mapsto \bar{A}_t$ is almost surely continuous, strictly increasing and by Lemma A.0.1 $\bar{A}_\infty = +\infty$. It follows that $(\bar{\tau}_t)_{t\geq 0}$ is well-defined and continuous.

Set $\delta = 1-\beta \in (0,2)$ and consider $(U_t^{(1-\beta)})_{t\geq 0}$ the process, defined above, associated to $(\bar{A}_t)_{t\geq 0}$ and $(\bar{\tau}_t)_{t\geq 0}$. Applying Lemma 2.2.2, with $a_\epsilon = \epsilon^{(\beta+1)/2}$, one obtains that $(\sqrt{\epsilon}V_t^\epsilon)_{t\geq 0} \overset{\mathcal{L}}{=} (\sqrt{\epsilon}V_{t/\epsilon})_{t\geq 0}$, where $(V_t^\epsilon)_{t\geq 0}$ is the process defined in Lemma 2.2.2. Then, it suffices to prove that $(\sqrt{\epsilon}V_t^\epsilon)_{t\geq 0} \overset{\mathcal{L}}{\Longrightarrow} (U_t^{(1-\beta)})_{t\geq 0}$.

As in step 2 of the proof of Lemma 2.2.1 *iv*), it suffices to prove that for all $T \geq 0$, $\sup_{[0,T]}\left|\sqrt{\epsilon}V_t^\epsilon - U_t^{(1-\beta)}\right| \overset{\mathbb{P}}{\longrightarrow} 0$, as $\epsilon \to 0$.

STEP 1: For all $T \geq 0$, $\lim_{\epsilon\to 0}\sup_{[0,T]}\left|\tau_t^\epsilon - \bar{\tau}_t\right| = 0$ almost surely.

Fix $T \geq 0$. Since $\sigma \geq c > 0$ and $\sigma(z) \underset{|z|\to\infty}{\sim} (\beta+1)|z|^{\beta/(\beta+1)}$, there exists $C > 0$ such that, for all $z \in \mathbb{R}$, $\sigma^{-2}(z) \leq C|z|^{-2\beta/(\beta+1)}$. Thus, by the dominated convergence theorem,

$$\sup_{[0,T]}\left|A_t^\epsilon - \bar{A}_t\right| \leq \int_0^T \left|\epsilon^{-\beta}\sigma^{-2}\left(\frac{W_s}{\epsilon^{(\beta+1)/2}}\right) - (\beta+1)^{-2}|W_s|^{-2\beta/(\beta+1)}\right|\mathrm{d}s \underset{\epsilon\to 0}{\longrightarrow} 0 \text{ almost surely,}$$

Indeed, since $\bar{A}_T$ is almost surely finite,

$$\left|\epsilon^{-\beta}\sigma^{-2}\left(\frac{W_s}{\epsilon^{(\beta+1)/2}}\right) - (\beta+1)^{-2}|W_s|^{-2\beta/(\beta+1)}\right| \leq (C+(\beta+1)^{-2})|W_s|^{-2\beta/(\beta+1)} \in L^1([0,T]).$$

Besides, $\epsilon^{-\beta}\sigma^{-2}\left(\dfrac{W_s}{\epsilon^{(\beta+1)/2}}\right) \underset{\epsilon\to 0}{\sim} (\beta+1)^{-2}|W_s|^{-2\beta/(\beta+1)}$. Then, using that $\bar{A}_\infty = +\infty$, it follows, by Lemma A.0.4, that $\lim_{\epsilon\to 0}\sup_{[0,T]}|\tau_t^\epsilon - \bar{\tau}_t| = 0$ almost surely.

STEP 2: For all $T \geq 0$, $\lim_{\epsilon\to 0}\sup_{[0,T]}\left|W_{\tau_t^\epsilon} - W_{\bar{\tau}_t}\right| = 0$ almost surely.

Fix $T \geq 0$. Since (τ_T^ϵ) converges and for all $t \in [0,T]$, $\tau_t^\epsilon \leq \tau_T^\epsilon$, there exists $\tilde{M}$ such that $\forall\epsilon > 0$ $\forall t \in [0,T]$, $\tau_t^\epsilon \leq \tilde{M}$ almost surely. Set $M = \max(\tilde{M}, \bar{\tau}_T)$. Fix $\eta > 0$, one can choose $\delta > 0$ such that

$$\forall x, y \in [0,M] \;\; |x-y| \leq \delta \Rightarrow |W_x - W_y| \leq \eta.$$

Almost surely, there exists ϵ_0 such that for all $\epsilon \leq \epsilon_0$, $\sup_{[0,T]}|\tau_t^\epsilon - \bar{\tau}_t| \leq \delta$, whence $\sup_{[0,T]}\left|W_{\tau_t^\epsilon} - W_{\bar{\tau}_t}\right| \leq \eta$.

STEP 3: For all $M > 0$, $\kappa_\epsilon(M) := \sup_{|z|\leq M}\left|\sqrt{\epsilon}h^{-1}(z/\epsilon^{(\beta+1)/2}) - \mathrm{sgn}(z)|z|^{1/(\beta+1)}\right| \underset{\epsilon\to 0}{\longrightarrow} 0$.

Fix $M > 0$. Define $\gamma : z \mapsto \dfrac{h^{-1}(z)}{\mathrm{sgn}(z)|z|^{1/(\beta+1)}} - 1$, with $\gamma(0) = -1$. Since h^{-1} is C^1, $h^{-1}(0) = 0$ and $h^{-1}(z) \underset{|z|\to\infty}{\sim} \mathrm{sgn}(z)|z|^{1/(\beta+1)}$, γ is continuous and $\lim_{|z|\to+\infty}\gamma(z) = 0$, hence γ is bounded. It follows that

$$\begin{aligned}\kappa_\epsilon(M) &= \sup_{|z|\leq M}\left|\gamma(z/\epsilon^{(\beta+1)/2})|z|^{1/(\beta+1)}\right| \leq \epsilon^{1/4}\parallel\gamma\parallel_\infty + M^{1/(\beta+1)}\sup_{|z|\geq\epsilon^{(\beta+1)/4}}\left|\gamma(z/\epsilon^{(\beta+1)/2})\right| \\ &\leq \epsilon^{1/4}\parallel\gamma\parallel_\infty + M^{1/(\beta+1)}\sup_{|z|\geq\epsilon^{-(\beta+1)/4}}|\gamma(z)| \underset{\epsilon\to 0}{\longrightarrow} 0.\end{aligned}$$

Step 4: Conclusion.
By step 2, $M_T = \sup_{[0,T]} \sup_{\epsilon\in(0,1)} \left|W_{\tau_t^\epsilon}\right|$ is a.s. finite. Thus, for $\epsilon \in (0,1)$, using steps 2 and 3,

$$\begin{aligned}\sup_{[0,T]}\left|\sqrt{\epsilon}V_t^\epsilon - U_t^{(1-\beta)}\right| &= \sup_{[0,T]}\left|\sqrt{\epsilon}h^{-1}(W_{\tau_t^\epsilon}/\epsilon^{(\beta+1)/2}) - \operatorname{sgn}(W_{\bar{\tau}_t})\left|W_{\bar{\tau}_t}\right|^{1/(\beta+1)}\right| \\ &\leq \kappa_\epsilon(M_T) + \sup_{[0,T]}\left|\operatorname{sgn}(W_{\tau_t^\epsilon})\left|W_{\tau_t^\epsilon}\right|^{1/(\beta+1)} - \operatorname{sgn}(W_{\bar{\tau}_t})\left|W_{\bar{\tau}_t}\right|^{1/(\beta+1)}\right| \underset{\epsilon\to 0}{\longrightarrow} 0 \text{ almost surely.}\end{aligned}$$

This ends the proof of Theorem 2.1.1 v). □

2.3.3 Case $\beta \in [1,5]$

Assume $\beta \in [1,5]$. One needs to state two lemma.

Lemma 2.3.1. *Fix $\alpha \in (0,2)$. Consider $(L_t^0)_{t\geq 0}$ the local time at 0 of $(W_t)_{t\geq 0}$ and its right-continuous generalized inverse $\tau_t = \inf\{u \geq 0, L_u^0 > t\}$. For $\eta > 0$, set $K_t^\eta := \int_0^t \operatorname{sgn}(W_s)\left|W_s\right|^{1/\alpha-2}\mathbb{1}_{\{|W_s|\geq\eta\}}\,\mathrm{d}s$. Then $(K_t^\eta)_{t\geq 0}$ converges a.s., as $\eta \to 0$, to a symmetric α-stable process $(K_t)_{t\geq 0}$, such that*

$$\mathbb{E}\left[e^{i\xi K_{\tau_t}}\right] = e^{-\kappa_\alpha t|\xi|^\alpha}, \quad \text{where } \kappa_\alpha = \frac{2^\alpha\pi\alpha^{2\alpha}}{2\alpha\Gamma(\alpha)^2\sin(\pi\alpha/2)}.$$

See [YB87].

Lemma 2.3.2. *Let $(L_t^0)_{t\geq 0}$ be the local time at 0 of $(W_t)_{t\geq 0}$. Consider $(K_t)_{t\geq 0}$ the process defined in the latter Lemma, with $\alpha = (\beta+1)/3$. For each $\epsilon > 0$, let $(A_t^\epsilon)_{t\geq 0}$ and $(H_t^\epsilon)_{t\geq 0}$ be the processes built in Lemma 2.2.2, with the choice $a_\epsilon = \dfrac{\epsilon}{(\beta+1)c_\beta}$, if $\beta \in (1,5]$ and $a_\epsilon = \epsilon\left|\log\epsilon\right|/2$, if $\beta = 1$ respectively. Then,*

i) For all $T > 0$, $\lim_{\epsilon\to 0}\sup_{[0,T]}\left|A_t^\epsilon - L_t^0\right| = 0$ almost surely.

ii) If $\beta \in (1,5)$, for all $T > 0$, $\lim_{\epsilon\to 0}\sup_{[0,T]}\left|\sqrt[\alpha]{\epsilon}H_t^\epsilon - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_t\right| = 0$ almost surely.

iii) If $\beta = 1$, for all $T > 0$, $\lim_{\epsilon\to 0}\sup_{[0,T]}\left|\left|\epsilon\log\epsilon\right|^{3/2}H_t^\epsilon - K_t/\sqrt{2}\right| = 0$ almost surely.

iv) If $\beta = 5$, for all $T > 0$, $\lim_{\epsilon\to 0}\sup_{[0,T]}\left|T_t^\epsilon - \sigma_5^2L_t^0\right| = 0$ a.s., where $T_t^\epsilon := \dfrac{\epsilon}{a_\epsilon^2\left|\log\epsilon\right|}\displaystyle\int_0^t \psi\left(\frac{W_s}{a_\epsilon}\right)\mathrm{d}s$.

Proof. Fix $T > 0$.

i) **Step 1: Assume first $\beta > 1$.**

Set $\gamma = (\beta+1)c_\beta$, recall that $a_\epsilon = \epsilon/\gamma$, so that $A_t^\epsilon = \dfrac{\gamma^2}{\epsilon}\displaystyle\int_0^t \sigma\left(\frac{\gamma W_s}{\epsilon}\right)^{-2}\mathrm{d}s$. By the occupation time formula, denoting L_t^x the local time at x of $(W_t)_{t\geq 0}$, one can write, for all $t \in [0,T]$,

$$A_t^\epsilon = \frac{\gamma^2}{\epsilon}\int_{\mathbb{R}}\sigma\left(\frac{\gamma x}{\epsilon}\right)^{-2}L_t^x\,\mathrm{d}x = \gamma\int_{\mathbb{R}}\sigma(y)^{-2}L_t^{\epsilon y/\gamma}\,\mathrm{d}y.$$

Moreover, by definition of c_β,

$$\int_{\mathbb{R}}\frac{\gamma}{\sigma^2(y)}\,\mathrm{d}y = \int_{\mathbb{R}}\frac{\gamma}{[h'(h^{-1}(y))]^2}\,\mathrm{d}y = \int_{\mathbb{R}}\frac{\gamma h'(x)}{h'(x)^2}\,\mathrm{d}x = \int_{\mathbb{R}}\frac{\gamma\vartheta(x)^\beta}{\beta+1}\,\mathrm{d}x = 1.$$

Consequently,

$$\sup_{[0,T]}\left|A_t^\epsilon - L_t^0\right| = \sup_{[0,T]}\left|\int_{\mathbb{R}}\sigma(y)^{-2}\gamma(L_t^{\epsilon y/\gamma} - L_t^0)\,\mathrm{d}y\right| \leq \gamma\int_{\mathbb{R}}\frac{\sup_{[0,T]}\left|L_t^{\epsilon y/\gamma} - L_t^0\right|}{\sigma(y)^2}\,\mathrm{d}y \underset{\epsilon\to 0}{\longrightarrow} 0 \text{ a.s.,}$$

by the dominated convergence theorem:

- for all $y \in \mathbb{R}$, $\sup_{[0,T]} \left| L_t^{\epsilon y/\gamma} - L_t^0 \right| \underset{\epsilon \to 0}{\longrightarrow} 0$ a.s., since $a \mapsto L_t^a$ is uniformly continuous in t on every compact set (see [Corollary 2.8 p. 226 in RY99]).
- for all $y \in \mathbb{R}$ and $\epsilon > 0$, $\sup_{[0,T]} \left| L_t^{\epsilon y/\gamma} - L_t^0 \right| \leq 2 \sup_{[0,T]\times\mathbb{R}} L_t^x < +\infty$ a.s. by Lemma A.0.5, and the fact that $1/\sigma^2$ is integrable.

Step 2: Assume $\beta = 1$.
Integrating the equivalent of σ yields, for all $x > 0$,

$$\int_{-x}^{x} \frac{\mathrm{d}y}{\sigma^2(y)} \underset{x\to\infty}{\sim} \frac{\log x}{2}. \tag{2.4}$$

Fix $\delta > 0$. One can write, for $t \in [0,T]$,

$$A_t^\epsilon = \frac{\epsilon}{a_\epsilon^2} \int_0^t \sigma\left(\frac{W_s}{a_\epsilon}\right)^{-2} \mathrm{d}s = \underbrace{\int_0^t \frac{\epsilon \mathbb{1}_{\{|W_s| \leq \delta\}}}{a_\epsilon^2 \sigma^2(W_s/a_\epsilon)} \mathrm{d}s}_{:=I_t^{\epsilon,\delta}} + \underbrace{\int_0^t \frac{\epsilon \mathbb{1}_{\{|W_s| > \delta\}}}{a_\epsilon^2 \sigma^2(W_s/a_\epsilon)} \mathrm{d}s}_{:=J_t^{\epsilon,\delta}}.$$

Since there exists $c > 0$ such that, for all $z \in \mathbb{R}$, $\sigma^2(z) \geq c(1+|z|)$, if $|W_s| > \delta$, then $\sigma^2(W_s/a_\epsilon) \geq c(1+\delta/a_\epsilon) \geq c\delta/a_\epsilon$ for ϵ small enough. Thus,

$$\sup_{[0,T]} \left| J_t^{\epsilon,\delta} \right| \leq \int_0^T \left| \frac{\epsilon \mathbb{1}_{\{|W_s| > \delta\}}}{a_\epsilon^2 \sigma^2(W_s/a_\epsilon)} \mathrm{d}s \right| \leq \int_0^T \frac{\epsilon}{a_\epsilon c\delta} \mathrm{d}s = \frac{T\epsilon}{a_\epsilon c\delta} \underset{\epsilon\to 0}{\longrightarrow} 0 \text{ almost surely.}$$

Using the occupation time formula one can write

$$I_t^{\epsilon,\delta} = \int_{-\delta}^{\delta} \frac{\epsilon L_t^x}{a_\epsilon^2 \sigma^2(x/a_\epsilon)} \mathrm{d}x = L_t^0 \underbrace{\int_{-\delta}^{\delta} \frac{\epsilon}{a_\epsilon^2 \sigma^2(x/a_\epsilon)} \mathrm{d}x}_{:=r_{\epsilon,\delta}} + \underbrace{\int_{-\delta}^{\delta} \frac{\epsilon (L_t^x - L_t^0)}{a_\epsilon^2 \sigma^2(x/a_\epsilon)} \mathrm{d}x}_{:=R_t^{\epsilon,\delta}}.$$

But, by (2.4) and the definition of a_ϵ,

$$r_{\epsilon,\delta} = \int_{-\delta/a_\epsilon}^{\delta/a_\epsilon} \frac{\epsilon}{a_\epsilon \sigma^2(y)} \mathrm{d}y \underset{\epsilon\to 0}{\sim} \frac{\epsilon \log(\delta/a_\epsilon)}{2a_\epsilon} \underset{\epsilon\to 0}{\longrightarrow} 1.$$

Using the decomposition of A_t^ϵ, one can write

$$\left| A_t^\epsilon - L_t^0 \right| \leq |r_{\epsilon,\delta} - 1| \, L_t^0 + \left| R_t^{\epsilon,\delta} \right| + \left| J_t^{\epsilon,\delta} \right|.$$

Thus,

$$\limsup_{\epsilon\to 0} \sup_{[0,T]} \left| A_t^\epsilon - L_t^0 \right| \leq \underbrace{\limsup_{\epsilon\to 0} |r_{\epsilon,\delta} - 1|}_{=0} \, L_T^0 + \limsup_{\epsilon\to 0} \sup_{[0,T]} \left| R_t^{\epsilon,\delta} \right| + \underbrace{\limsup_{\epsilon\to 0} \sup_{[0,T]} \left| J_t^{\epsilon,\delta} \right|}_{=0}.$$

Moreover,

$$\sup_{[0,T]} \left| R_t^{\epsilon,\delta} \right| \leq r_{\epsilon,\delta} \sup_{[0,T]\times[-\delta,\delta]} \left| L_t^x - L_t^0 \right|.$$

So, by [Corollary 1.8 p.226 in RY99],

$$\limsup_{\epsilon\to 0} \sup_{[0,T]} \left| A_t^\epsilon - L_t^0 \right| \leq \sup_{[0,T]\times[-\delta,\delta]} \left| L_t^x - L_t^0 \right| \underset{\delta\to 0}{\longrightarrow} 0 \text{ almost surely.}$$

ii) **Step 1: The process $(K_t^\eta)_{t\geq 0}$, defined in Lemma 2.3.1, converges almost surely uniformly on $[0,T]$, as $\eta \to 0$, to $K_t = \int_{\mathbb{R}} \operatorname{sgn}(x)\,|x|^{(1-2\beta)/(\beta+1)}\,(L_t^x - L_t^0 \mathbb{1}_{\{|x|\leq 1\}})\,\mathrm{d}x$.**
Assume $\beta \in (1,5)$. Set $\gamma = (\beta+1)c_\beta$. Since $\alpha = (\beta+1)/3$, $1/\alpha - 2 = (1-2\beta)/(\beta+1)$. With the notation of Lemma 2.3.1, from the occupation time formula and symmetry it follows that, for all $t \geq 0$,

$$K_t^\eta = \int_{\mathbb{R}} \operatorname{sgn}(x)\,|x|^{(1-2\beta)/(\beta+1)}\,\mathbb{1}_{\{|x|\geq\eta\}} L_t^x\,\mathrm{d}x = \int_{\mathbb{R}} \operatorname{sgn}(x)\,|x|^{(1-2\beta)/(\beta+1)}\,\mathbb{1}_{\{|x|\geq\eta\}} (L_t^x - L_t^0 \mathbb{1}_{\{|x|\leq 1\}})\,\mathrm{d}x.$$

Fix $\vartheta \in (0, \frac{1}{2})$. One has

$$M_{\vartheta,T} := \sup_{[0,T]\times\mathbb{R}} (|x| \wedge 1)^{-\vartheta} \left|L_t^x - L_t^0 \mathbb{1}_{\{|x|\leq 1\}}\right| \leq \sup_{[0,T]\times[-1,1]} |x|^{-\vartheta} \left|L_t^x - L_t^0\right| + \sup_{[0,T]\times\mathbb{R}} L_t^x < +\infty \text{ a.s.},$$

by Lemma A.0.5 and the fact that $x \mapsto L_t^x$ is ϑ-Hölder uniformly in t on every compact set (see [Corollary 1.8 p.226 in RY99]).
Set $\tilde{K}_t = \int_{\mathbb{R}} \operatorname{sgn}(x)\,|x|^{(1-2\beta)/(\beta+1)}\,(L_t^x - L_t^0 \mathbb{1}_{\{|x|\leq 1\}})\,\mathrm{d}x$. One can write, for $\eta \leq 1$,

$$\begin{aligned}\sup_{[0,T]} \left|K_t^\eta - \tilde{K}_t\right| &= \sup_{[0,T]} \left|\int_{\mathbb{R}} \operatorname{sgn}(x)\,|x|^{(1-2\beta)/(\beta+1)}\,\mathbb{1}_{\{|x|<\eta\}} (L_t^x - L_t^0 \mathbb{1}_{\{|x|\leq 1\}})\,\mathrm{d}x\right| \\ &\leq M_{\vartheta,T} \int_{-\eta}^{\eta} |x|^{\vartheta + (1-2\beta)/(\beta+1)}\,\mathrm{d}x,\end{aligned}$$

where $\vartheta \in (0, \frac{1}{2})$ is chosen to be $\vartheta = \frac{1}{2} - \delta/2$, for $\delta = \frac{1-2\beta}{\beta+1} + \frac{3}{2} \in (0,1)$ (because $\beta \in (1,5)$). One can conclude by the dominated convergence theorem. Moreover, by Lemma 2.3.1, $(K_t^\eta)_{t\geq 0}$ converges pointwise to $(K_t)_{t\geq 0}$, hence $K_t = \tilde{K}_t$, for all $t \geq 0$.
Step 2: Conclusion.
Observe that $\sqrt[\alpha]{\epsilon}\, a_\epsilon^{-2} = \gamma^2 \epsilon^{(1-2\beta)/(\beta+1)}$. Thus, by the occupation time formula and the fact that ϕ is odd, it follows that, for $t \in [0,T]$,

$$\begin{aligned}\sqrt[\alpha]{\epsilon}\, H_t^\epsilon &= \gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \int_0^t \phi\left(\frac{\gamma W_s}{\epsilon}\right) \mathrm{d}s = \gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \int_{\mathbb{R}} \phi\left(\frac{\gamma x}{\epsilon}\right) L_t^x\,\mathrm{d}x \\ &= \gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \int_{\mathbb{R}} \phi\left(\frac{\gamma x}{\epsilon}\right) (L_t^x - L_t^0 \mathbb{1}_{\{|x|\leq 1\}})\,\mathrm{d}x.\end{aligned}$$

Consequently, for any $\vartheta \in (0, \frac{1}{2})$,

$$\begin{aligned}&\sup_{[0,T]} \left|\sqrt[\alpha]{\epsilon}\, H_t^\epsilon - (\beta+1)^{1/\alpha - 2} c_\beta^{1/\alpha} K_t\right| \\ &\leq \int_{\mathbb{R}} \left|\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi\left(\frac{\gamma x}{\epsilon}\right) - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} \operatorname{sgn}(x)\,|x|^{(1-2\beta)/(\beta+1)}\right| \sup_{[0,T]} \left|L_t^x - L_t^0 \mathbb{1}_{\{|x|\leq 1\}}\right| \mathrm{d}x \\ &\leq M_{\vartheta,T} \int_{\mathbb{R}} \left|\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi\left(\frac{\gamma x}{\epsilon}\right) - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} \operatorname{sgn}(x)\,|x|^{(1-2\beta)/(\beta+1)}\right| (|x| \wedge 1)^\vartheta\,\mathrm{d}x.\end{aligned}$$

In order to apply the dominated convergence theorem the following two facts need to be checked:

- With the equivalent for ϕ, for all $x \in \mathbb{R}$,
$$\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi\left(\frac{\gamma x}{\epsilon}\right) \underset{\epsilon\to 0}{\sim} (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} \operatorname{sgn}(x)\,|x|^{(1-2\beta)/(\beta+1)}, \text{ by definition of } \gamma.$$
- Using that for all $z \in \mathbb{R}$, $|\phi(z)| \leq C\,|z|^{(1-2\beta)/(\beta+1)}$, one has, for $\epsilon > 0$,
$$\left|\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi\left(\frac{\gamma x}{\epsilon}\right)\right| (|x| \wedge 1)^\vartheta \leq \begin{cases} \tilde{C}\,|x|^{\vartheta + (1-2\beta)/(\beta+1)} & \text{if } |x| \leq 1 \\ \tilde{\tilde{C}}\,|x|^{(1-2\beta)/(\beta+1)} & \text{if } |x| > 1,\end{cases}$$
which is an integrable function for $\beta \in (2,5)$. Here $\vartheta \in (0, \frac{1}{2})$ is chosen as in the previous step.

For $\beta \in (1,2]$, one has to proceed quite differently. Indeed, the last function is integrable on $\{|x| \leq 1\}$, so

$$\sup_{[0,T]} \left| \int_{|x|\leq 1} \left(\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi\left(\frac{\gamma x}{\epsilon}\right) - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} \operatorname{sgn}(x) |x|^{(1-2\beta)/(\beta+1)} \right) (L_t^x - L_t^0)\, \mathrm{d}x \right| \underset{\epsilon\to 0}{\longrightarrow} 0.$$

It remains to show that

$$\sup_{[0,T]} \left| \int_{|x|>1} \left(\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi\left(\frac{\gamma x}{\epsilon}\right) - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} \operatorname{sgn}(x) |x|^{(1-2\beta)/(\beta+1)} \right) L_t^x\, \mathrm{d}x \right| \underset{\epsilon\to 0}{\longrightarrow} 0.$$

Fix $t \in [0,T]$, by the occupation-time formula, one can write

$$\begin{aligned}&\int_{|x|>1} \left(\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi\left(\frac{\gamma x}{\epsilon}\right) - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} \operatorname{sgn}(x) |x|^{(1-2\beta)/(\beta+1)} \right) L_t^x\, \mathrm{d}x \\ &\quad = \int_0^t \left(\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi\left(\frac{\gamma W_s}{\epsilon}\right) - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} \operatorname{sgn}(W_s) |W_s|^{(1-2\beta)/(\beta+1)} \right) \mathbb{1}_{\{|W_s|>1\}}\, \mathrm{d}s.\end{aligned}$$

Hence,

$$\begin{aligned}&\sup_{[0,T]} \left| \int_{|x|>1} \left(\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi\left(\frac{\gamma x}{\epsilon}\right) - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} \operatorname{sgn}(x) |x|^{(1-2\beta)/(\beta+1)} \right) L_t^x\, \mathrm{d}x \right| \\ &\quad \leq \int_0^T \left(\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi\left(\frac{\gamma W_s}{\epsilon}\right) - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} \operatorname{sgn}(W_s) |W_s|^{(1-2\beta)/(\beta+1)} \right) \mathbb{1}_{\{|W_s|>1\}}\, \mathrm{d}s.\end{aligned}$$

Now the dominated convergence theorem is applied:

- As before, for all $s \in [0,T]$,

$$\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi\left(\frac{\gamma W_s}{\epsilon}\right) \underset{\epsilon\to 0}{\sim} (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} \operatorname{sgn}(W_s) |W_s|^{(1-2\beta)/(\beta+1)}.$$

- For all $\epsilon > 0$ and $s \in [0,T]$,

$$\left| \gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi\left(\frac{\gamma W_s}{\epsilon}\right) \right| \mathbb{1}_{\{|W_s|>1\}} \leq \tilde{C} |W_s|^{(1-2\beta)/(\beta+1)} \mathbb{1}_{\{|W_s|>1\}} \leq \tilde{C} \in L^1([0,T]),$$

  since $\frac{1-2\beta}{\beta+1} < 0$.

This concludes the proof.

iii) Assume $\beta = 1$ and set $a_\epsilon = \epsilon |\log \epsilon| /2$. With the notations of Lemma 2.3.1, it follows, by the occupation time formula, that, for $t \geq 0$,

$$K_t^\eta = \int_{\mathbb{R}} \operatorname{sgn}(x) |x|^{-1/2} \mathbb{1}_{\{|x|\geq\eta\}} L_t^x\, \mathrm{d}x.$$

Setting $\tilde{K}_t := \int_0^t \operatorname{sgn}(W_s) |W_s|^{-1/2} \mathrm{d}s = \int_{\mathbb{R}} \operatorname{sgn}(x) |x|^{-1/2} L_t^x\, \mathrm{d}x$, one gets

$$\left| \tilde{K}_t - K_t^\eta \right| \leq \int_{\mathbb{R}} |x|^{-1/2} L_t^x \mathbb{1}_{\{|x|<\eta\}}\, \mathrm{d}x \leq \sup_{\mathbb{R}} L_t^x \int_{-\eta}^{\eta} \frac{1}{\sqrt{x}}\, \mathrm{d}x \underset{\eta\to 0}{\longrightarrow} 0.$$

Using Lemma 2.3.1, one obtains that $K_t = \tilde{K}_t = \int_0^t \operatorname{sgn}(W_s) |W_s|^{-1/2} \mathrm{d}s$.

Besides, $|\epsilon \log \epsilon|^{3/2} H_t^\epsilon = 4 |\epsilon \log \epsilon|^{-1/2} \int_0^t \phi\left(\frac{2W_s}{\epsilon |\log \epsilon|}\right) \mathrm{d}s$. This yields

$$\sup_{[0,T]} \left| |\epsilon \log \epsilon|^{3/2} H_t^\epsilon - K_t/\sqrt{2} \right| \leq \int_0^T \left| 4 |\epsilon \log \epsilon|^{-1/2} \phi\left(\frac{2W_s}{\epsilon |\log \epsilon|}\right) - \frac{\operatorname{sgn}(W_s) |W_s|^{-1/2}}{\sqrt{2}} \right| \mathrm{d}s \underset{\epsilon\to 0}{\longrightarrow} 0,$$

by the dominated convergence theorem:

- for all $s \in [0,T]$, $4\,|\epsilon \log \epsilon|^{-1/2}\, \phi\left(\frac{2W_s}{\epsilon\,|\log \epsilon|}\right) \underset{\epsilon\to 0}{\longrightarrow} \frac{\operatorname{sgn}(W_s)\,|W_s|^{-1/2}}{\sqrt{2}}$, using the equivalent of ϕ.
- Since for all $z \in \mathbb{R}$, $|\phi(z)| \le C\,|z|^{-1/2}$, then, for all $s \in [0,T]$ and $\epsilon > 0$,

$$\left|4\,|\epsilon \log \epsilon|^{-1/2}\, \phi\left(\frac{2W_s}{\epsilon\,|\log \epsilon|}\right)\right| \le \tilde{C}\,|W_s|^{-1/2} \in L^1([0,T]),$$

  by Remark 2.3.2.

iv) Assume $\beta = 5$, the proof is very similar to the first point with $\beta = 1$. Set $\gamma = 6c_5$. Taking into account the information known about ψ, it follows from the integration of the equivalent, that, for all $x > 0$,

$$\int_{-x}^{x} \psi(z)\,\mathrm{d}z \underset{x\to\infty}{\sim} \frac{2\log x}{81}. \tag{2.5}$$

Besides, one can write

$$T_t^\epsilon = \int_0^t \frac{\gamma^2}{\epsilon\,|\log \epsilon|}\psi\left(\frac{\gamma W_s}{\epsilon}\right)\mathrm{d}s = \underbrace{\int_0^t \frac{\gamma^2}{\epsilon\,|\log \epsilon|}\psi\left(\frac{\gamma W_s}{\epsilon}\right)\mathbb{1}_{\{|W_s|\le\delta\}}\,\mathrm{d}s}_{:=\tilde{I}_t^{\epsilon,\delta}} + \underbrace{\int_0^t \frac{\gamma^2}{\epsilon\,|\log \epsilon|}\psi\left(\frac{\gamma W_s}{\epsilon}\right)\mathbb{1}_{\{|W_s|>\delta\}}\,\mathrm{d}s}_{:=\tilde{J}_t^{\epsilon,\delta}}.$$

Since there exists $\tilde{c} > 0$ such that, for all $z \in \mathbb{R}$, $\psi(z) \le \frac{\tilde{c}}{|z|}$, if $|W_s| > \delta$, then $\psi(\gamma W_s/\epsilon) \le \frac{\tilde{c}\epsilon}{\gamma\delta}$. Thus,

$$\sup_{[0,T]}\left|\tilde{J}_t^{\epsilon,\delta}\right| \le \int_0^T \left|\frac{\gamma^2\mathbb{1}_{\{|W_s|>\delta\}}}{\epsilon\,|\log \epsilon|}\psi\left(\frac{\gamma W_s}{\epsilon}\right)\mathrm{d}s\right| \le \frac{T\gamma\tilde{c}}{|\log \epsilon|\,\delta} \underset{\epsilon\to 0}{\longrightarrow} 0 \text{ almost surely.}$$

One can then use the occupation time formula to write

$$\tilde{I}_t^{\epsilon,\delta} = \int_{-\delta}^{\delta} \frac{\gamma^2}{\epsilon\,|\log \epsilon|}\psi\left(\frac{\gamma x}{\epsilon}\right)L_t^x\,\mathrm{d}x = L_t^0 \underbrace{\int_{-\delta}^{\delta} \frac{\gamma^2}{\epsilon\,|\log \epsilon|}\psi\left(\frac{\gamma x}{\epsilon}\right)\mathrm{d}x}_{:=\tilde{r}_{\epsilon,\delta}} + \underbrace{\int_{-\delta}^{\delta} \frac{\gamma^2(L_t^x - L_t^0)}{\epsilon\,|\log \epsilon|}\psi\left(\frac{\gamma x}{\epsilon}\right)\mathrm{d}x}_{:=\tilde{R}_t^{\epsilon,\delta}}.$$

But, by (2.5),

$$\tilde{r}_{\epsilon,\delta} = \int_{-\gamma\delta/\epsilon}^{\gamma\delta/\epsilon} \frac{\gamma}{|\log \epsilon|}\psi(y)\,\mathrm{d}y \underset{\epsilon\to 0}{\sim} \frac{2\gamma\log(\gamma\delta/\epsilon)}{81\,|\log \epsilon|} \underset{\epsilon\to 0}{\longrightarrow} \sigma_5^2.$$

By the decomposition of T_t^ϵ, one can write

$$\left|T_t^\epsilon - \sigma_5^2 L_t^0\right| \le \left|\tilde{r}_{\epsilon,\delta} - \sigma_5^2\right| L_t^0 + \left|\tilde{R}_t^{\epsilon,\delta}\right| + \left|\tilde{J}_t^{\epsilon,\delta}\right|.$$

Thus,

$$\limsup_{\epsilon\to 0}\sup_{[0,T]}\left|T_t^\epsilon - \sigma_5^2 L_t^0\right| \le \underbrace{\limsup_{\epsilon\to 0}\left|\tilde{r}_{\epsilon,\delta} - \sigma_5^2\right|}_{=0} L_T^0 + \limsup_{\epsilon\to 0}\sup_{[0,T]}\left|\tilde{R}_t^{\epsilon,\delta}\right| + \underbrace{\limsup_{\epsilon\to 0}\sup_{[0,T]}\left|\tilde{J}_t^{\epsilon,\delta}\right|}_{=0}.$$

Moreover,

$$\sup_{[0,T]}\left|\tilde{R}_t^{\epsilon,\delta}\right| \le \tilde{r}_{\epsilon,\delta}\sup_{[0,T]\times[-\delta,\delta]}\left|L_t^x - L_t^0\right|.$$

So

$$\limsup_{\epsilon\to 0}\sup_{[0,T]}\left|T_t^\epsilon - \sigma_5^2 L_t^0\right| \le \sup_{[0,T]\times[-\delta,\delta]}\left|L_t^x - L_t^0\right| \underset{\delta\to 0}{\longrightarrow} 0 \text{ a.s., by [Corollary 1.8 p.226 in RY99].}$$

□

Proof of Theorem 2.1.1 ii) − iv). Assume $\beta \in [1,5]$. Denote by $(L_t^0)_{t\geq 0}$ the local time of $(W_t)_{t\geq 0}$ and set $\tau_t = \inf\{u \geq 0, L_u^0 > t\}$ its generalized inverse. Keep the notations of Lemma 2.3.1 with $\alpha = (\beta+1)/3$ and Lemma 2.2.2 with

$$a_\epsilon = \begin{cases} \epsilon/[(\beta+1)c_\beta] & \text{if } \beta \in (1,5], \\ \epsilon\,|\log \epsilon| & \text{if } \beta = 1. \end{cases}$$

ii) Assume $\beta = 5$, as seen in Remark 2.3.1, it suffices to show that, for each $t \geq 0$,

$$\frac{\epsilon}{|\log \epsilon|}\int_0^{t/\epsilon} g'(V_s)^2\,\mathrm{d}s = \frac{1}{|\log \epsilon|}\int_0^t g'(V_{s/\epsilon})^2\,\mathrm{d}s \xrightarrow{\mathbb{P}} \sigma_5^2 t, \text{ as } \epsilon \to 0.$$

Thanks to Lemma 2.2.2, it is equivalent to show that, for each $t \geq 0$, $J_t^\epsilon := \dfrac{1}{|\log \epsilon|}\displaystyle\int_0^t g'(V_s^\epsilon)^2\,\mathrm{d}s \xrightarrow{\mathbb{P}} \sigma_5^2 t$. For all $t \geq 0$,

$$J_t^\epsilon = \int_0^t \frac{g'(h^{-1}(W_{\tau_s^\epsilon}/a_\epsilon))^2}{|\log \epsilon|}\,\mathrm{d}s = \int_0^{\tau_t^\epsilon} \frac{\epsilon g'(h^{-1}(W_u/a_\epsilon))^2}{a_\epsilon^2\,|\log \epsilon|\,\sigma(W_u/a_\epsilon)^2}\,\mathrm{d}u = \frac{\epsilon}{a_\epsilon^2\,|\log \epsilon|}\int_0^{\tau_t^\epsilon} \psi\left(\frac{W_u}{a_\epsilon}\right)\mathrm{d}u = T_{\tau_t^\epsilon}^\epsilon.$$

One knows, by Lemma 2.3.2, that, for all $T > 0$, $\sup_{[0,T]}\left|A_t^\epsilon - L_t^0\right| \underset{\epsilon\to 0}{\longrightarrow} 0$ almost surely. Since $(\tau_t)_{t\geq 0}$ has no fixed times of jumps (see [Theorem 8 p. 114 in Ber98]), it follows from Lemma A.0.4, that, for all $t \geq 0$, $\tau_t^\epsilon \underset{\epsilon\to 0}{\longrightarrow} \tau_t$ almost surely. Moreover, for all $t \geq 0$,

$$\left|J_t^\epsilon - \sigma_5^2 t\right| \leq \left|T_{\tau_t^\epsilon}^\epsilon - \sigma_5^2 L_{\tau_t^\epsilon}^0\right| + \sigma_5^2\left|L_{\tau_t^\epsilon}^0 - L_{\tau_t}^0\right| + \sigma_5^2\left|L_{\tau_t}^0 - t\right|.$$

Hence, using again Lemma 2.3.2 and the fact that $T := \sup_{\epsilon\in(0,1)} \tau_t^\epsilon$ is almost surley finite, the first term tends to 0. For the second term, one can use the fact that $\tau_t^\epsilon \underset{\epsilon\to 0}{\longrightarrow} \tau_t$ a.s. and the almost sure continuity of the local time. The last term is equal to 0 almost surely.

iii) Assume $\beta \in (1,5)$. By Lemma 2.2.1, one can assume again that $X_0 = V_0 = 0$ so that, by Lemma 2.2.2, $(X_{t/\epsilon})_{t\geq 0} \overset{\mathcal{L}}{=} (H_{\tau_t^\epsilon})_{t\geq 0}$. Thanks to Lemma 2.3.1, $S_t^{(\alpha)} := \sigma_\beta^{-1}(\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_{\tau_t}$ is a symmetric α-stable process with $\mathbb{E}[\exp(iuS_t^{(\alpha)})] = \exp(-t\,|u|^\alpha)$. Hence, as already seen, it suffices to prove that, for each $t \geq 0$, $\delta_t(\epsilon) = \left|\sqrt[\alpha]{\epsilon}H_{\tau_t^\epsilon}^\epsilon - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_{\tau_t}\right| \underset{\epsilon\to 0}{\longrightarrow} 0$ almost surely. Fix $t \geq 0$. As previously, $\tau_t^\epsilon \underset{\epsilon\to 0}{\longrightarrow} \tau_t$ a.s. and by Lemma 2.3.2, for all $T > 0$, $\limsup_{\epsilon\to 0}\sup_{[0,T]}\left|\sqrt[\alpha]{\epsilon}H_t^\epsilon - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_t\right| = 0$ almost surely. Hence,

$$\delta_t(\epsilon) \leq \left|\sqrt[\alpha]{\epsilon}H_{\tau_t^\epsilon}^\epsilon - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_{\tau_t^\epsilon}\right| + (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}\left|K_{\tau_t^\epsilon} - K_{\tau_t}\right|.$$

The first term tends to 0 almost surely, since $T := \sup_{\epsilon\in(0,1)} \tau_t^\epsilon$ is almost surely finite. The second term tends to 0 by the continuity of $(K_t)_{t\geq 0}$. One gets, as previously, the convergence.

iv) Assume $\beta = 1$. Using the same argument as before, $S_t^{(2/3)} := (\sqrt{2}\sigma_1)^{-1}K_{\tau_t}$ is a symmetric stable pocess of index $\frac{2}{3}$ with $\mathbb{E}[\exp(iuS_t^{(2/3)})] = \exp(-t\,|u|^{2/3})$. Thus, it suffices to prove that, for all $t \geq 0$, $\tilde{\delta}_t(\epsilon) := \left||\epsilon\log\epsilon|^{3/2}H_{\tau_t^\epsilon}^\epsilon - K_{\tau_t}/\sqrt{2}\right| \underset{\epsilon\to 0}{\longrightarrow} 0$ almost surely. But, for all $t \geq 0$,

$$\tilde{\delta}_t(\epsilon) \leq \left||\epsilon\log\epsilon|^{3/2}H_{\tau_t^\epsilon}^\epsilon - K_{\tau_t^\epsilon}/\sqrt{2}\right| + \left|K_{\tau_t^\epsilon} - K_{\tau_t}\right|/\sqrt{2}.$$

By Lemma 2.3.2 again and the continuity of $(K_t)_{t\geq 0}$, $\tilde{\delta}_t(\epsilon)$ converges to 0 almost surely, as $\epsilon \to 0$.

□

2.4 Proof of Corollary 2.1.2

Assume $\beta > 1$ and consider $(V_t, X_t)_{t\geq 0}$ the solution to (2.1), associated to some Brownian motion $(B_t)_{t\geq 0}$, and starting from some initial condition (V_0, X_0). Define, for $t \geq 0$, $\mathcal{F}_t := \sigma(X_0, V_0, B_s, s \leq t)$. Theorem 2.1.1 yields $(v_\epsilon^{(\beta)} X_{t/\epsilon})_{t\geq 0} \overset{\text{f.d}}{\Longrightarrow} (X_t^{(\beta)})_{t\geq 0}$, where the speed $v_\epsilon^{(\beta)} \underset{\epsilon\to 0}{\longrightarrow} 0$ and the limiting process $(X_t^{(\beta)})_{t\geq 0}$ are those appearing in Theorem 2.1.1. Fix $t \geq 0$. One has to show that $(v_\epsilon^{(\beta)} X_{t/\epsilon}, V_{t/\epsilon}) \overset{\mathcal{L}}{\Longrightarrow} (X_t^{(\beta)}, \tilde{V})$. By [Theorem 4.29 p. 78 in Kal02], the density of regular functions and the independence of $\tilde{V}$, it is sufficient to show that, for all $\phi \in C_b^1(\mathbb{R})$ and $\psi \in C_b(\mathbb{R})$,

$$\Delta_\epsilon := \left| \mathbb{E}\left[\phi(v_\epsilon^{(\beta)} X_{t/\epsilon})\psi(V_{t/\epsilon})\right] - \mathbb{E}\left[\phi(X_t^{(\beta)})\right] \int_{\mathbb{R}} \psi \, \mathrm{d}\mu_\beta \right| \underset{\epsilon\to 0}{\longrightarrow} 0.$$

Fix $\phi \in C_b^1(\mathbb{R})$ and $\psi \in C_b(\mathbb{R})$. Call $\mu_\beta(\psi) := \int_{\mathbb{R}} \psi \, \mathrm{d}\mu_\beta$.

Step 1: For all $\boldsymbol{h \in (0,t)}$, $\boldsymbol{\delta_\epsilon := \mathbb{E}\left[\left|\mathbb{E}\left[\psi(V_{t/\epsilon}) | \mathcal{F}_{(t-h)/\epsilon}\right] - \mu_\beta(\psi)\right|\right] \underset{\epsilon\to 0}{\longrightarrow} 0}$.

Fix $h \in (0, t)$. The usual notation $P_t\psi(v) = \mathbb{E}_v[\psi(V_t)]$ and $\|\cdot\|_{TV}$, for the total variation norm, is used. Let $\tilde{V}$ be a μ_β-distributed random variable independent of $X^{(\beta)}$, such that $\mathbb{P}(\tilde{V} \neq V_{(t-h)/\epsilon}) = \|\mathcal{L}(V_{(t-h)/\epsilon}) - \mu_\beta\|_{TV}$. By Markov's property,

$$\begin{aligned}
\delta_\epsilon &= \mathbb{E}\left[\left|\mathbb{E}_{V_{(t-h)/\epsilon}}\left[\psi(V_{h/\epsilon})\right] - \mu_\beta(\psi)\right|\right] = \mathbb{E}\left[\left|P_{h/\epsilon}\psi(V_{(t-h)/\epsilon}) - \mu_\beta(\psi)\right|\right] \\
&\leq \underbrace{\mathbb{E}\left[\left|P_{h/\epsilon}\psi(V_{(t-h)/\epsilon}) - P_{h/\epsilon}\psi(\tilde{V})\right|\right]}_{:=\delta_\epsilon^1} + \underbrace{\mathbb{E}\left[\left|P_{h/\epsilon}\psi(\tilde{V}) - \mu_\beta(\psi)\right|\right]}_{:=\delta_\epsilon^2}.
\end{aligned}$$

Besides, $\delta_\epsilon^1 \leq 2\|\psi\|_\infty \mathbb{P}(\tilde{V} \neq V_{(t-h)/\epsilon}) = 2\|\psi\|_\infty \|\mathcal{L}(V_{(t-h)/\epsilon}) - \mu_\beta\|_{TV} \underset{\epsilon\to 0}{\longrightarrow} 0$, since μ_β is the invariant measure of $(V_t)_{t\geq 0}$. For the same reason, using the dominated convergence theorem, $\delta_\epsilon^2 \underset{\epsilon\to 0}{\longrightarrow} 0$.

Step 2: Conclusion.

Fix $h \in (0, t)$. One can write $\Delta_\epsilon \leq \Delta_{\epsilon,h}^1 + \Delta_{\epsilon,h}^2 + \Delta_{\epsilon,h}^3 + \Delta_h^4$, where

$$\begin{aligned}
\Delta_{\epsilon,h}^1 &:= \left| \mathbb{E}\left[\phi(v_\epsilon^{(\beta)} X_{t/\epsilon})\psi(V_{t/\epsilon})\right] - \mathbb{E}\left[\phi(v_\epsilon^{(\beta)} X_{(t-h)/\epsilon})\psi(V_{t/\epsilon})\right] \right| \\
\Delta_{\epsilon,h}^2 &:= \left| \mathbb{E}\left[\phi(v_\epsilon^{(\beta)} X_{(t-h)/\epsilon})\psi(V_{t/\epsilon})\right] - \mathbb{E}\left[\phi(v_\epsilon^{(\beta)} X_{(t-h)/\epsilon})\right] \mu_\beta(\psi) \right| \\
\Delta_{\epsilon,h}^3 &:= \left| \mathbb{E}\left[\phi(v_\epsilon^{(\beta)} X_{(t-h)/\epsilon})\right] \mu_\beta(\psi) - \mathbb{E}\left[\phi(X_{t-h}^{(\beta)})\right] \mu_\beta(\psi) \right| \\
\Delta_h^4 &:= \left| \mathbb{E}\left[\phi(X_{t-h}^{(\beta)})\right] \mu_\beta(\psi) - \mathbb{E}\left[\phi(X_t^{(\beta)})\right] \mu_\beta(\psi) \right|.
\end{aligned}$$

By Theorem 2.1.1, $\Delta_{\epsilon,h}^3 \underset{\epsilon\to 0}{\longrightarrow} 0$ and by step 1 $\Delta_{\epsilon,h}^2 \leq \|\phi\|_\infty \delta_\epsilon \underset{\epsilon\to 0}{\longrightarrow} 0$. Besides, set $C := \|\psi\|_\infty(\|\phi\|_\infty + \|\phi'\|_{\infty,K})$, where the compact set K is chosen such that for ϵ and h small enough, $(v_\epsilon^{(\beta)} X_{t/\epsilon}, v_\epsilon^{(\beta)} X_{(t-h)/\epsilon}) \in K^2$. Then, by the dominated convergence theorem,

$$\limsup_{\epsilon\to 0} \Delta_{\epsilon,h}^1 \leq C \limsup_{\epsilon\to 0} \mathbb{E}\left[\left|v_\epsilon^{(\beta)} X_{t/\epsilon} - v_\epsilon^{(\beta)} X_{(t-h)/\epsilon}\right| \wedge 1\right] = C\mathbb{E}\left[\left|X_t^{(\beta)} - X_{(t-h)}^{(\beta)}\right| \wedge 1\right] \underset{h\to 0}{\longrightarrow} 0$$

Likewise, $\Delta_h^4 \underset{h\to 0}{\longrightarrow} 0$.

This ends the proof of Corollary 2.1.2. □

Chapter 3

Asymptotic behaviour of solution of a time-inhomogeneous kinetic equation

3.1 Introduction and main result

One can focus now on time-inhomogeneous kinetic equation. Consider the stochastic kinetic model:

$$\begin{aligned} V_t &= V_0 + B_t + \rho \int_0^t \frac{\operatorname{sgn}(V_s)\,|V_s|^\alpha}{s^\beta}\,\mathrm{d}s, \\ X_t &= X_0 + \int_0^t V_s\,\mathrm{d}s, \end{aligned} \tag{3.1}$$

where $\alpha, \beta, \rho \in \mathbb{R}$ and $(B_t)_{t\geq 0}$ is a Brownian motion. In [Off12], the asymptotic behaviour of the velocity process is studied. The interest is now on the asymptotic behaviour of the position process. Thanks to [Propositions 2.3.2 and 2.3.6 in Off12], there exists a pathwise unique strong solution $(V_t)_{t\geq 0}$ defined up to the explosion time, which is almost surely finite, and it is a Markov process.

Theorem 3.1.1. *Consider $\rho < 0$, $\alpha \geq 0$, and $\beta \in \mathbb{R}$ such that $2\beta - (\alpha+1) > 0$. Let $(V_t, X_t)_{t\geq 0}$ be a solution of* (3.1). *Then, as ϵ converges to* 0,

$$(\epsilon^{3/2} X_{t/\epsilon})_{t\geq 1} \overset{f.d}{\Longrightarrow} (\beta_{t^3/3})_{t\geq 1}.$$

Here $(\beta_t)_{t\geq 0}$ is a Brownian motion.

Remark 3.1.1. If one tries to adapt the proof of Theorem 2.1.1 *i)* naively, one is led to find a solution to

$$\frac{\partial g}{\partial v}(s,v)\rho\frac{\operatorname{sgn}(v)\,|v|^\alpha}{s^\beta} + \frac{\partial g}{\partial s}(s,v) + \frac{1}{2}\frac{\partial^2 g}{\partial^2 v}(s,v) = -v.$$

But this PDE is ill-posed. Thus one has to proceed quite differently, this is due to the time-dependance of the stochastic differential equation satisfied by the velocity process.

3.2 Study of a changed-of-time process

Following the idea used in [Off12], one can perform first a change of time in (3.1). Denoting by $\phi_e : t \mapsto e^t$ the exponential change of time, the exponential scaling transformation is then given by $\Phi_e(\omega) : s \in \mathbb{R}^+ \mapsto \frac{\omega_{e^s}}{e^{s/2}}$, for $\omega \in \Omega$. Set $V^{(e)} := \Phi_e(V)$ and $X_t^{(e)} := \int_0^t V_s^{(e)}\,\mathrm{d}s$, for $t \geq 0$.

The process $V^{(e)}$ satisfies the equation

$$\mathrm{d}V_s^{(e)} = \mathrm{d}W_s - \frac{V_s^{(e)}}{2}\,\mathrm{d}s + \rho e^{(\frac{\alpha+1}{2}-\beta)s}\operatorname{sgn}(V_s^{(e)})\left|V_s^{(e)}\right|^\alpha \mathrm{d}s, \tag{3.2}$$

where $(W_t)_{t\geq 0}$ is a Brownian motion.

Remark 3.2.1. Observe the time-inhomogeneous part: leaving out the last term, it yields the equation of the Ornstein-Ulhenbeck process:

$$\mathrm{d}U_s = \mathrm{d}W_s - \frac{U_s}{2}\,\mathrm{d}s.$$

The last term in (3.2) seems to be negligible.

Lemma 3.2.1. *If $\rho < 0$, $\alpha > -1$ and $2\beta - (\alpha+1) > 0$, then, for all $t \geq 0$,*

$$\lim_{\epsilon\to 0} V_{t/\epsilon}^{(e)} \stackrel{\mathcal{L}}{=} \mathcal{N}(0,1).$$

Moreover, almost surely, $\limsup_{t\to\infty} \frac{\left|V_t^{(e)}\right|}{\sqrt{2\ln(t)}} = 1.$

Proof. The convergence in distribution comes from the proof of [Theorem 2.4.6 (2.4.15) in Off12], Besides, by [2.4.15 in Off12], $\limsup_{t\to\infty} \frac{V_t^{(e)}}{\sqrt{2\ln(t)}} = 1$. But $(-V_t^{(e)})_{t\geq 0}$ satisfies the same equation as $(V_t^{(e)})_{t\geq 0}$. So, one can adapt the proof of [Off12] in order to find that

$$1 = \limsup_{t\to\infty} \frac{-V_t^{(e)}}{\sqrt{2\ln(t)}} = -\liminf_{t\to\infty} \frac{V_t^{(e)}}{\sqrt{2\ln(t)}}.$$

So that $\liminf_{t\to\infty} \frac{V_t^{(e)}}{\sqrt{2\ln(t)}} = -1$. The conclusion follows. □

Lemma 3.2.2. *If $\rho < 0$, $\alpha \geq 0$ and $2\beta - (\alpha+1) > 0$, then, as ϵ tends to 0,*

$$(\sqrt{\epsilon} X_{t/\epsilon}^{(e)})_{t\geq 0} \stackrel{f.d}{\Longrightarrow} (2W_t)_{t\geq 0}.$$

Proof. If $g \in C^2$, by Itô's formula,

$$\mathrm{d}g(V_s^{(e)}) = g'(V_s^{(e)})\,\mathrm{d}W_s + \left(\frac{1}{2}g''(V_s^{(e)}) - g'(V_s^{(e)})\frac{V_s^{(e)}}{2}\right)\mathrm{d}s + g'(V_s^{(e)})\rho e^{(\frac{\alpha+1}{2}-\beta)s}\,\mathrm{sgn}(V_s^{(e)})\left|V_s^{(e)}\right|^{\alpha}\mathrm{d}s.$$

One would like the second term in the right-hand side to be equal to $-V_s^{(e)}$. Taking $g : v \mapsto 2v$ yields

$$2\,\mathrm{d}V_s^{(e)} = 2\,\mathrm{d}W_s - V_s^{(e)}\,\mathrm{d}s + 2\rho e^{(\frac{\alpha+1}{2}-\beta)s}\,\mathrm{sgn}(V_s^{(e)})\left|V_s^{(e)}\right|^{\alpha}\mathrm{d}s.$$

It follows that

$$X_t^{(e)} = 2V_0^{(e)} - 2V_t^{(e)} + 2W_t + \int_0^t 2\rho e^{(\frac{\alpha+1}{2}-\beta)s}\,\mathrm{sgn}(V_s^{(e)})\left|V_s^{(e)}\right|^{\alpha}\mathrm{d}s. \tag{3.3}$$

By Lemma 3.2.1, $\lim_{\epsilon\to 0} V_{t/\epsilon}^{(e)} \stackrel{\mathcal{L}}{=} \mathcal{N}(0,1)$. For $\epsilon > 0$, setting $v_\epsilon \underset{\epsilon\to 0}{\longrightarrow} 0$ for the rate, one can write

$$v_\epsilon X_{t/\epsilon}^{(e)} = \underbrace{2v_\epsilon V_0^{(e)} - 2v_\epsilon V_{t/\epsilon}^{(e)}}_{\substack{\hookrightarrow 0 \text{ a.s.} \\ \text{by Slutsky lemma}}} + 2v_\epsilon W_{t/\epsilon} + v_\epsilon \int_0^{t/\epsilon} 2\rho e^{(\frac{\alpha+1}{2}-\beta)s}\,\mathrm{sgn}(V_s^{(e)})\left|V_s^{(e)}\right|^{\alpha}\mathrm{d}s.$$

With $v_\epsilon = \sqrt{\epsilon}$ it becomes

$$\sqrt{\epsilon} X_{t/\epsilon}^{(e)} = \underbrace{2\sqrt{\epsilon} V_0^{(e)} - 2\sqrt{\epsilon} V_{t/\epsilon}^{(e)}}_{\hookrightarrow 0 \text{ a.s.}} + \underbrace{2\sqrt{\epsilon} W_{t/\epsilon}}_{\stackrel{\mathcal{L}}{=} 2W_t} + \sqrt{\epsilon}\int_0^{t/\epsilon} 2\rho e^{(\frac{\alpha+1}{2}-\beta)s}\,\mathrm{sgn}(V_s^{(e)})\left|V_s^{(e)}\right|^{\alpha}\mathrm{d}s. \tag{3.4}$$

The dominated convergence theorem can be applied to the last term:

- For $s \geq 0$, $\sqrt{\epsilon}\mathbb{1}_{[0,t/\epsilon]}2\rho e^{(\frac{\alpha+1}{2}-\beta)s}\operatorname{sgn}(V_s^{(e)})\left|V_s^{(e)}\right|^\alpha \underset{\epsilon\to 0}{\longrightarrow} 0$.
- For all $\epsilon < 1$ and $s \geq 0$,

$$\left|\sqrt{\epsilon}\mathbb{1}_{[0,t/\epsilon]}2\rho e^{(\frac{\alpha+1}{2}-\beta)s}\operatorname{sgn}(V_s^{(e)})\left|V_s^{(e)}\right|^\alpha\right| \leq \mathbb{1}_{\mathbb{R}^+}2\left|\rho\right| e^{(\frac{\alpha+1}{2}-\beta)s}\left|V_s^{(e)}\right|^\alpha \in L^1,$$

since $\alpha \geq 0$ and $\limsup_{t\to\infty} \frac{\left|V_t^{(e)}\right|}{\sqrt{2\ln(t)}} = 1$ a.s. (see Lemma 3.2.1).

One concludes as in the proof of Theorem 2.1.1 *i*). □

The Figure 3.1 illustrates this convergence.

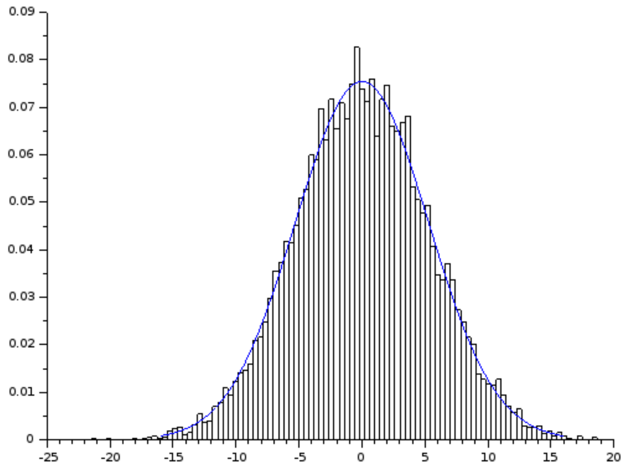

Figure 3.1: Distribution of $\sqrt{\epsilon}X_{t/\epsilon}^{(e)}$ and overlay with a Gaussian random variable $\mathcal{N}(0, 4t)$, with $t = 7$ and $\epsilon = 10^{-4}$.

Moreover, one can give a speed for the convergence:

Lemma 3.2.3. *If* $\rho < 0$, $\alpha \geq 0$ *and* $2\beta - (\alpha+1) > 0$, *then,* $\limsup_{t\to\infty} \frac{X_t^{(e)}}{2\sqrt{2t\ln(\ln(t))}} = 1$ *almost surely.*

Proof. By the law of iterated logarithm for the Brownian motion,

$$\limsup_{t\to\infty} \frac{W_t}{\sqrt{2t\ln(\ln(t))}} = 1 \text{ almost surely.}$$

From (3.3), it follows that a.s.

$$\begin{aligned}
\limsup_{t\to+\infty} \frac{X_t^{(e)}}{2\sqrt{2t\ln(\ln(t))}} &= \limsup_{t\to+\infty} \frac{2V_0^{(e)}}{\sqrt{2t\ln(\ln(t))}} - \limsup_{t\to+\infty} \frac{V_t^{(e)}}{\sqrt{2t\ln(\ln(t))}} + \limsup_{t\to+\infty} \frac{W_t}{\sqrt{2t\ln(\ln(t))}} \\
&\quad + \rho \limsup_{t\to+\infty} \frac{1}{\sqrt{2t\ln(\ln(t))}} \int_0^t e^{(\frac{\alpha+1}{2}-\beta)s}\operatorname{sgn}(V_s^{(e)})\left|V_s^{(e)}\right|^\alpha \mathrm{d}s \\
&= \limsup_{t\to+\infty} \frac{V_t^{(e)}}{\sqrt{2\ln(t)}} \frac{\sqrt{2\ln(t)}}{\sqrt{2t\ln(\ln(t))}} + 1 \\
&\quad + \rho \limsup_{t\to+\infty} \frac{1}{\sqrt{2t\ln(\ln(t))}} \int_0^t e^{(\frac{\alpha+1}{2}-\beta)s}\operatorname{sgn}(V_s^{(e)})\left|V_s^{(e)}\right|^\alpha \mathrm{d}s \\
&= 1 + \rho \limsup_{t\to+\infty} \frac{1}{\sqrt{2t\ln(\ln(t))}} \int_0^t e^{(\frac{\alpha+1}{2}-\beta)s}\operatorname{sgn}(V_s^{(e)})\left|V_s^{(e)}\right|^\alpha \mathrm{d}s.
\end{aligned}$$

One can shows that $\limsup_{t\to+\infty} \frac{1}{\sqrt{2t\ln(\ln(t))}} \int_0^t e^{(\frac{\alpha+1}{2}-\beta)s} \operatorname{sgn}(V_s^{(e)}) \left|V_s^{(e)}\right|^\alpha \mathrm{d}s = 0$ almost surely. Indeed, since $\limsup_{t\to\infty} \frac{\left|V_t^{(e)}\right|}{\sqrt{2\ln(t)}} = 1$ a.s.,then, for all $\epsilon > 0$, there exists $A > 1$ such that $\frac{\left|V_t^{(e)}\right|}{\sqrt{2\ln(t)}} < 1+\epsilon$. It follows that

$$\limsup_{t\to+\infty} \frac{1}{\sqrt{2t\ln(\ln(t))}} \int_0^t e^{(\frac{\alpha+1}{2}-\beta)s} \operatorname{sgn}(V_s^{(e)}) \left|V_s^{(e)}\right|^\alpha \mathrm{d}s = \limsup_{t\to+\infty} \frac{1}{\sqrt{2t\ln(\ln(t))}} \int_A^t e^{(\frac{\alpha+1}{2}-\beta)s} \operatorname{sgn}(V_s^{(e)}) \left|V_s^{(e)}\right|^\alpha \mathrm{d}s.$$

Then, for all $t \geq A$,

$$\begin{aligned}
\frac{1}{\sqrt{2t\ln(\ln(t))}} \left| \int_A^t e^{(\frac{\alpha+1}{2}-\beta)s} \operatorname{sgn}(V_s^{(e)}) \left|V_s^{(e)}\right|^\alpha \mathrm{d}s \right| &\leq \frac{1}{\sqrt{2t\ln(\ln(t))}} \int_A^t e^{(\frac{\alpha+1}{2}-\beta)s} \left|V_s^{(e)}\right|^\alpha \mathrm{d}s \\
&\leq \frac{\sqrt{2\ln(t)}^\alpha}{\sqrt{2t\ln(\ln(t))}} \int_A^t e^{(\frac{\alpha+1}{2}-\beta)s} \frac{\left|V_s^{(e)}\right|^\alpha}{\sqrt{2\ln(s)}^\alpha} \left(\frac{\ln(s)}{\ln(t)}\right)^{\alpha/2} \mathrm{d}s \\
&\leq (1+\epsilon)^\alpha \frac{\sqrt{2\ln(t)}^\alpha}{\sqrt{2t\ln(\ln(t))}} \int_A^t e^{(\frac{\alpha+1}{2}-\beta)s} \mathrm{d}s, \text{ since } \alpha \geq 0 \\
&\leq \frac{(1+\epsilon)^\alpha}{\frac{\alpha+1}{2}-\beta} \frac{\sqrt{2\ln(t)}^\alpha}{\sqrt{2t\ln(\ln(t))}} \left[e^{(\frac{\alpha+1}{2}-\beta)t} - e^{(\frac{\alpha+1}{2}-\beta)A} \right] \underset{t\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

because $\frac{\alpha+1}{2} - \beta < 0$. This concludes the proof. □

In fact, it is possible to find a formula for $V^{(e)}$:

Lemma 3.2.4. *For all $t \geq 0$,*

$$V_t^{(e)} = V_0^{(e)} e^{-t/2} + \int_0^t e^{-(t-s)/2} \mathrm{d}W_s + \rho \int_0^t e^{-(t-s)/2} e^{(\frac{\alpha+1}{2}-\beta)s} \operatorname{sgn}(V_s^{(e)}) \left|V_s^{(e)}\right|^\alpha \mathrm{d}s. \tag{3.5}$$

Proof. Writing differently (3.2), one has

$$\mathrm{d}V_s^{(e)} + \frac{V_s^{(e)}}{2} \mathrm{d}s = \mathrm{d}W_s + \rho e^{(\frac{\alpha+1}{2}-\beta)s} \operatorname{sgn}(V_s^{(e)}) \left|V_s^{(e)}\right|^\alpha \mathrm{d}s$$

This can be solved, using the method of variation of parameters. Indeed, $V^{(e)}$ can be written as $V_t^{(e)} = C_t e^{-t/2}$, for $t \geq 0$. Here C is a process that must be determined. It satisfies

$$\mathrm{d}C_s = e^{s/2} \mathrm{d}W_s + \rho e^{(\frac{\alpha+1}{2}-\beta)s} e^{s/2} \operatorname{sgn}(V_s^{(e)}) \left|V_s^{(e)}\right|^\alpha \mathrm{d}s.$$

Hence, for all $t \geq 0$,

$$C_t = V_0^{(e)} + \int_0^t e^{s/2} \mathrm{d}W_s + \int_0^t \rho e^{(\frac{\alpha+1}{2}-\beta)s} e^{s/2} \operatorname{sgn}(V_s^{(e)}) \left|V_s^{(e)}\right|^\alpha \mathrm{d}s.$$

This ends the proof. □

Remark 3.2.2. $V_t^{(e)}$ can be written as $V_t^{(e)} = \tilde{V}_t^{(e)} + U_t$, where $\tilde{V}^{(e)}$ is an Ornstein Ulhenbeck process and $U_t := \int_0^t \rho e^{-(t-s)/2} e^{(\frac{\alpha+1}{2}-\beta)s} \operatorname{sgn}(V_s^{(e)}) \left|V_s^{(e)}\right|^\alpha \mathrm{d}s$. This is useful for the simulation.

3.3 Proof of Theorem 3.1.1

In this section, $\rho < 0$, $\alpha \geq 0$, and $\beta \in \mathbb{R}$ are such that $2\beta - (\alpha + 1) > 0$.
The goal is know to find the asymptotic behaviour of $(X_t)_{t\geq 0}$.

Step 1: Write $(X_t)_{t\geq 0}$ as a function of $(X_t^{(e)})_{t\geq 0}$.
Firstly, for all $t \geq 0$,

$$\begin{aligned} X_t^{(e)} &= \int_0^t V_s^{(e)}\,\mathrm{d}s = \int_1^{e^t} \frac{V_u}{u^{3/2}}\,\mathrm{d}u \overset{IBP}{=} \frac{X_{e^t}}{e^{3t/2}} - X_1 + \frac{3}{2}\int_1^{e^t} \frac{X_s}{s^{5/2}} ds \\ &= X_{e^t} e^{-3t/2} - X_1 + \frac{3}{2}\int_0^t X_{e^u} e^{-3u/2}\,\mathrm{d}u. \end{aligned} \tag{3.6}$$

Setting $v_\epsilon \underset{\epsilon\to 0}{\longrightarrow} 0$ for the rate, this yields, for $\epsilon > 0$,

$$\begin{aligned} v_\epsilon X_{t/\epsilon}^{(e)} &= X_{e^{t/\epsilon}} v_\epsilon e^{-3t/2\epsilon} - v_\epsilon X_1 + \frac{3v_\epsilon}{2}\int_0^{t/\epsilon} X_{e^u} e^{-3u/2}\,\mathrm{d}u \\ &= X_{e^{t/\epsilon}} v_\epsilon e^{-3t/2\epsilon} - v_\epsilon X_1 + \frac{3v_\epsilon}{2\epsilon}\int_0^{t} X_{e^{s/\epsilon}} e^{-3s/2\epsilon}\,\mathrm{d}s \end{aligned}$$

But the behaviour of the third term of the right-hand side is unknown. However, observe that (3.6) may be written, setting $G : t \mapsto \int_0^t X_{e^u} e^{-3u/2}\,\mathrm{d}u$, as

$$G'(t) + \frac{3}{2}G(t) = X_t^{(e)} + X_1, \ G(0) = 0.$$

This ODE can be solved :

$$G : t \mapsto e^{-3t/2}\int_0^t \left(X_s^{(e)} + X_1\right) e^{3s/2}\,\mathrm{d}s = e^{-3t/2}\int_0^t X_s^{(e)} e^{3s/2}\,\mathrm{d}s + \frac{2}{3}X_1(1 - e^{-3t/2}).$$

Hence, using the two equality of G', one obtains that, for all $t \geq 0$,

$$X_{e^t} e^{-3t/2} = X_t^{(e)} - \frac{3}{2}e^{-3t/2}\int_0^t X_s^{(e)} e^{3s/2}\,\mathrm{d}s + X_1 e^{-3t/2}.$$

This yields

$$\begin{aligned} X_{e^{t/\epsilon}} e^{-3t/2\epsilon} &= X_{t/\epsilon}^{(e)} - \frac{3}{2}e^{-3t/2\epsilon}\int_0^{t/\epsilon} X_s^{(e)} e^{3s/2}\,\mathrm{d}s + X_1 e^{-3t/2\epsilon} \\ &= X_{t/\epsilon}^{(e)} - \frac{3}{2\epsilon}e^{-3t/2\epsilon}\int_0^{t} X_{u/\epsilon}^{(e)} e^{3u/2\epsilon}\,\mathrm{d}u + X_1 e^{-3t/2\epsilon}. \end{aligned}$$

Step 2: Study of the middle term.
Since for $u \geq 0$, $X_{u/\epsilon}^{(e)} = \frac{1}{\epsilon}\int_0^u V_{s/\epsilon}^{(e)}\,\mathrm{d}s$, one gets

$$\begin{aligned} \frac{3}{2\epsilon}e^{-3t/2\epsilon}\int_0^t X_{u/\epsilon}^{(e)} e^{3u/2\epsilon}\,\mathrm{d}u &= \frac{3}{2\epsilon^2}e^{-3t/2\epsilon}\int_0^t\int_0^u V_{s/\epsilon}^{(e)}\,\mathrm{d}s e^{3u/2\epsilon}\,\mathrm{d}u = \frac{3}{2\epsilon^2}e^{-3t/2\epsilon}\int_0^t V_{s/\epsilon}^{(e)}\int_s^t e^{3u/2\epsilon}\,\mathrm{d}u\,\mathrm{d}s \\ &= \frac{3}{2\epsilon^2}e^{-3t/2\epsilon}\int_0^t V_{s/\epsilon}^{(e)}\frac{2\epsilon}{3}\left(e^{3t/2\epsilon} - e^{3s/2\epsilon}\right)\mathrm{d}s \\ &= \frac{1}{\epsilon}\left(\int_0^t V_{s/\epsilon}^{(e)}\,\mathrm{d}s - e^{-3t/2\epsilon}\int_0^t V_{s/\epsilon}^{(e)} e^{3s/2\epsilon}\,\mathrm{d}s\right) \\ &= X_{t/\epsilon}^{(e)} - \frac{1}{\epsilon}e^{-3t/2\epsilon}\int_0^t V_{s/\epsilon}^{(e)} e^{3s/2\epsilon}\,\mathrm{d}s. \end{aligned}$$

It yields

$$X_{e^{t/\epsilon}} e^{-3t/2\epsilon} = e^{-3t/2\epsilon}\int_0^{t/\epsilon} V_s^{(e)} e^{3s/2}\,\mathrm{d}s + X_1 e^{-3t/2\epsilon}. \tag{3.7}$$

Moreover, applying Itô's formula,

$$\begin{aligned}
V^{(e)}_{t/\epsilon}e^{3t/2\epsilon} &= V^{(e)}_0 + \frac{3}{2}\int_0^{t/\epsilon} V^{(e)}_s e^{3s/2}\,\mathrm{d}s + \int_0^{t/\epsilon} e^{3s/2}\,\mathrm{d}V^{(e)}_s \\
&= V^{(e)}_0 + \int_0^{t/\epsilon} V^{(e)}_s e^{3s/2}\,\mathrm{d}s + \int_0^{t/\epsilon} e^{3s/2}\,\mathrm{d}W_s + \int_0^{t/\epsilon} e^{3s/2}\rho e^{(\frac{\alpha+1}{2}-\beta)s}\operatorname{sgn}(V^{(e)}_s)\left|V^{(e)}_s\right|^\alpha \mathrm{d}s.
\end{aligned}$$

Hence,

$$\begin{aligned}
X_{e^{t/\epsilon}}e^{-3t/2\epsilon} = e^{-3t/2\epsilon}(X_1 - V^{(e)}_0) + V^{(e)}_{t/\epsilon} &- e^{-3t/2\epsilon}\int_0^{t/\epsilon} e^{3s/2}\,\mathrm{d}W_s \\
&- e^{-3t/2\epsilon}\int_0^{t/\epsilon} e^{3s/2}\rho e^{(\frac{\alpha+1}{2}-\beta)s}\operatorname{sgn}(V^{(e)}_s)\left|V^{(e)}_s\right|^\alpha \mathrm{d}s.
\end{aligned}$$

It follows, for all $u \geq 1$,

$$\epsilon^{3/2}X_{u/\epsilon} = \epsilon^{3/2}(X_1 - V^{(e)}_0) + u^{3/2}V^{(e)}_{\ln(\frac{u}{\epsilon})} - \epsilon^{3/2}\int_0^{\ln(\frac{u}{\epsilon})} e^{3s/2}\,\mathrm{d}W_s - \epsilon^{3/2}\int_0^{\ln(\frac{u}{\epsilon})} e^{3s/2}\rho e^{(\frac{\alpha+1}{2}-\beta)s}\operatorname{sgn}(V^{(e)}_s)\left|V^{(e)}_s\right|^\alpha \mathrm{d}s. \tag{3.8}$$

Step 3: Letting $\epsilon \to 0$.

The first and the last terms converge to 0 a.s. by the dominated convergence theorem:

- For all $s \geq 0$, $\epsilon^{3/2}\mathbb{1}_{[0,\ln(u/\epsilon)]}e^{3s/2}\rho e^{(\frac{\alpha+1}{2}-\beta)s}\operatorname{sgn}(V^{(e)}_s)\left|V^{(e)}_s\right|^\alpha \underset{\epsilon\to 0}{\longrightarrow} 0$ a.s.
- For all $\epsilon > 0$ and $s \geq 0$,

$$\begin{aligned}
\left|\epsilon^{3/2}\mathbb{1}_{[0,\ln(u/\epsilon)]}e^{3s/2}\rho e^{(\frac{\alpha+1}{2}-\beta)s}\operatorname{sgn}(V^{(e)}_s)\left|V^{(e)}_s\right|^\alpha\right| &= u^{3/2}\underbrace{\mathbb{1}_{[0,\ln(u/\epsilon)]}e^{\frac{-3(\ln(u/\epsilon)-s)}{2}}}_{\leq 1}|\rho|\,e^{(\frac{\alpha+1}{2}-\beta)s}\left|V^{(e)}_s\right|^\alpha \\
&\leq u^{3/2}|\rho|\,e^{(\frac{\alpha+1}{2}-\beta)s}\left|V^{(e)}_s\right|^\alpha \mathbb{1}_{\mathbb{R}^+}(s) \in L^1,\ \text{as seen before.}
\end{aligned}$$

Then, one can write, for all $u \geq 1$,

$$\epsilon^{3/2}X_{u/\epsilon} = Y^\epsilon_u + u^{3/2}V^{(e)}_{\ln(\frac{u}{\epsilon})} - \epsilon^{3/2}\int_0^{\ln(\frac{u}{\epsilon})} e^{3s/2}\,\mathrm{d}W_s,$$

where $Y^\epsilon_u \underset{\epsilon\to 0}{\longrightarrow} 0$ almost surely. Using Lemma 3.2.4, it becomes

$$\begin{aligned}
\epsilon^{3/2}X_{u/\epsilon} = Y^\epsilon_u + \sqrt{\epsilon}uV^{(e)}_0 &+ \int_0^{\ln(\frac{u}{\epsilon})}\left[u^{3/2}e^{-\frac{\ln(\frac{u}{\epsilon})-s}{2}} - \epsilon^{3/2}e^{3s/2}\right]\mathrm{d}W_s \\
&+ u^{3/2}\int_0^{\ln(\frac{u}{\epsilon})} \rho e^{(\frac{\alpha+1}{2}-\beta)s}e^{-\frac{\ln(\frac{u}{\epsilon})-s}{2}}\operatorname{sgn}(V^{(e)}_s)\left|V^{(e)}_s\right|^\alpha \mathrm{d}s.
\end{aligned}$$

The last term converges to 0 a.s. by the dominated convergence theorem:

- For all $s \geq 0$, $u^{3/2}\mathbb{1}_{[0,\ln(u/\epsilon)]}\rho e^{(\frac{\alpha+1}{2}-\beta)s}e^{s/2}\sqrt{\frac{\epsilon}{u}}\operatorname{sgn}(V^{(e)}_s)\left|V^{(e)}_s\right|^\alpha \underset{\epsilon\to 0}{\longrightarrow} 0$ a.s.
- For all $\epsilon > 0$ and $s \geq 0$,

$$\begin{aligned}
\left|u^{3/2}\rho e^{(\frac{\alpha+1}{2}-\beta)s}\mathbb{1}_{[0,\ln(u/\epsilon)]}e^{-\frac{\ln(\frac{u}{\epsilon})-s}{2}}\operatorname{sgn}(V^{(e)}_s)\left|V^{(e)}_s\right|^\alpha\right| &= u^{3/2}|\rho|\,e^{(\frac{\alpha+1}{2}-\beta)s}\underbrace{\mathbb{1}_{[0,\ln(u/\epsilon)]}e^{-\frac{\ln(\frac{u}{\epsilon})-s}{2}}}_{\leq 1}\left|V^{(e)}_s\right|^\alpha \\
&\leq u^{3/2}|\rho|\,e^{(\frac{\alpha+1}{2}-\beta)s}\left|V^{(e)}_s\right|^\alpha \mathbb{1}_{\mathbb{R}^+}(s) \in L^1,
\end{aligned}$$

  as already seen.

It follows that, for all $u \geq 1$,

$$\epsilon^{3/2} X_{u/\epsilon} = Z_u^\epsilon + \underbrace{\int_0^{\ln(\frac{u}{\epsilon})} \left[u^{3/2} e^{-\frac{\ln(\frac{u}{\epsilon})-s}{2}} - \epsilon^{3/2} e^{3s/2} \right] \mathrm{d}W_s}_{:=\epsilon^{3/2} M_{\ln(\frac{u}{\epsilon})}},$$

where Z_u^ϵ converges to 0 almost surely and $M_t := \int_0^t \left(e^{3t/2} e^{-\frac{t-s}{2}} - e^{3s/2} \right) \mathrm{d}W_s$. The process $(M_t)_{t\geq 0}$ is a continuous local martingale, vanishing at 0, with bracket $\langle M, M \rangle_t = \dfrac{(e^t - 1)^3}{3}$. Hence $\langle M, M \rangle_\infty = \infty$, so by Dambis-Dubins-Schwarz theorem ([Theorem 1.6 page 181 in RY99]), there exists a Brownian motion $(\beta_t)_{t\geq 0}$ such that $M_t = \beta_{\frac{(e^t-1)^3}{3}}$. One can then write

$$\epsilon^{3/2} X_{u/\epsilon} = Z_u^\epsilon + \epsilon^{3/2} \beta_{\frac{(u/\epsilon - 1)^3}{3}} \overset{\mathcal{L}}{=} Z_u^\epsilon + \beta_{\frac{(u-\epsilon)^3}{3}}.$$

Then it suffices to apply Lemma A.0.2, as in the proof of Theorem 2.1.1 *i*).

This ends the proof of Theorem 3.1.1. □

This convergence can be illustrated, using the equality (3.7) and Remark 3.2.2, by Figures 3.2 and 3.3, depending on which way the simulation is done. See Appendix B for details.

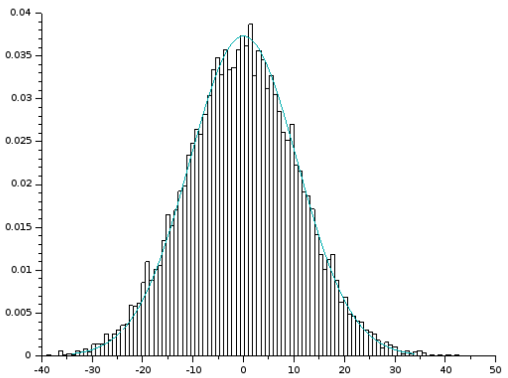

Figure 3.2: Distribution of $\epsilon^{3/2} X_{t/\epsilon} \approx \epsilon^{3/2} \int_0^{\log(t/\epsilon)} \tilde{V}_s^{(e)} e^{3s/2} \,\mathrm{d}u$ and overlay with a Gaussian random variable $\mathcal{N}(0, t^3/3)$, for $t = 7$ and $\epsilon = 10^{-4}$.

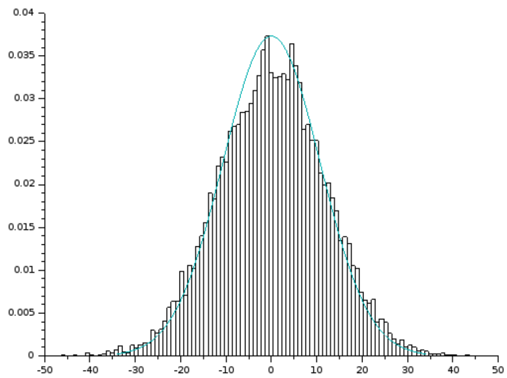

Figure 3.3: Distribution of $\epsilon^{3/2} X_{t/\epsilon} \approx \epsilon^{3/2} \int_0^{\log(t/\epsilon)} (\tilde{V}_s^{(e)} + U_s) e^{3s/2} \,\mathrm{d}u$ and overlay with a Gaussian random variable $\mathcal{N}(0, t^3/3)$, for $t = 7$ and $\epsilon = 10^{-4}$.

Appendix A

Technical results

Lemma A.0.1. *Let $G : \mathbb{R} \to \mathbb{R}$ be a positive function which could be zero only at isolated points. Consider $(W_t)_{t\geq 0}$ a real Brownian motion. Then $\int_0^{+\infty} G(W_s)\,\mathrm{d}s = +\infty$ almost surely.*

Proof. Choose $x \in \mathbb{R}$ and $\epsilon > 0$ such that $G(]x-2\epsilon, x+2\epsilon[) \subset \mathbb{R}^{+*}$. Then if $W_s \in [x-\epsilon, x+\epsilon]$, $G(W_s) \geq \inf_{[x-\epsilon,x+\epsilon]} G > 0$. Define the stopping times $\tau_0 = \inf\{t \geq 0, W_t \in]x-\epsilon, x+\epsilon[\}$, $\sigma_0 = \inf\{t \geq \tau_0, W_t \notin]x-\epsilon, x+\epsilon[\}$ and for $i \geq 0$, $\tau_{i+1} := \inf\{t \geq \sigma_i, W_s \in]x-\epsilon, x+\epsilon[\}$ and $\sigma_{i+1} = \inf\{t \geq \tau_{i+1}, W_t \notin]x-\epsilon, x+\epsilon[\}$. Then

$$\int_0^{+\infty} G(W_s)\,\mathrm{d}s \geq \sum_i \int_{\tau_i}^{\sigma_i} G(W_s)\,\mathrm{d}s.$$

But, thanks to strong Markov property, $Y_i := \int_{\tau_i}^{\sigma_i} G(W_s)\,\mathrm{d}s$ are i.i.d. random variables with positive expectation. Hence, by the law of large numbers, $\sum_i \int_{\tau_i}^{\sigma_i} G(W_s)\,\mathrm{d}s = +\infty$ almost surely. □

Lemma A.0.2. *Let S be a separable metric space. Let $(Y_n, Z_n) \in S \times S$ be a sequence of processes on S such that $Y_n \Longrightarrow Y$ (for the convergence in law in S) and $\rho(Y_n, Z_n) \overset{\mathbb{P}}{\to} 0$ where ρ is a metric on S. Then $Z_n \Longrightarrow Y$.*

Proof. See [Theorem 3.1 p. 27 in Bil99]. □

Lemma A.0.3. *If $Y_\epsilon \overset{\mathcal{L}}{\Longrightarrow} Y$ in $C := C([0, +\infty[)$, and the sequence of functions $(g_\epsilon)_{\epsilon>0}$ converges uniformly to some continuous function g. Then $g_\epsilon(Y_\epsilon) \overset{\mathcal{L}}{\Longrightarrow} g(Y)$.*

Proof. Let h be a bounded and uniformly continuous function, one has to show that $\mathbb{E}[h \circ g_\epsilon(Y_\epsilon)] \underset{\epsilon\to 0}{\longrightarrow} \mathbb{E}[h \circ g(Y)]$. One can write

$$\mathbb{E}[h \circ g_\epsilon(Y_\epsilon)] = \mathbb{E}[h \circ g_\epsilon(Y_\epsilon) - h \circ g(Y_\epsilon)] + \mathbb{E}[h \circ g(Y_\epsilon)].$$

The second term converges to $\mathbb{E}[h \circ g(Y)]$ since $(Y_\epsilon)_{\epsilon>0}$ converges in distribution towards Y and $h \circ g$ is continuous and bounded. It remains to show that $\mathbb{E}[h \circ g_\epsilon(Y_\epsilon) - h \circ g(Y_\epsilon)] \underset{\epsilon\to 0}{\longrightarrow} 0$. h is uniformly continuous and $(g_\epsilon)_{\epsilon>0}$ converges uniformly to g so $(h \circ g_\epsilon)_{\epsilon>0}$ converges uniformly to $h \circ g$. Then $|\mathbb{E}[h \circ g_\epsilon(Y_\epsilon) - h \circ g(Y_\epsilon)]| \leq \| h \circ g_\epsilon - h \circ g \|_\infty \underset{\epsilon\to 0}{\longrightarrow} 0$.

□

Proposition A.0.1. *Let $\mathcal{M} \subset \mathcal{M}_1(C([0,T]))$ be tight. Then $\lim_{\delta\to 0} \sup_{\mu\in\mathcal{M}} \mu(\{x | w_\delta(x) \geq \eta\}) = 0$, where $w_\delta(f) := \sup\{|f(t) - f(s);\ s,t \in [0,T],\ |t-s| \leq \delta|\}$, for every $f \in C([0,T])$..*

Proof. See [Theorem 7.3 p. 82 in Bil99]. □

Lemma A.0.4. *Let, for $n \geq 1$, $(a_t^n)_{t\geq 0}$ be a continuous increasing bijective function from $\mathbb{R}^+$ to itself, as well as its inverse $(r_t^n)_{t\geq 0}$.*

1. *Assume that* $(a^n_t)_{t\geq 0}$ *converges pointwise to some function* $(a_t)_{t\geq 0}$ *such that* $\lim\limits_{t\to+\infty} a_t = +\infty$*, call* $r_t = \inf\{u \geq 0, a_u > t\}$*, its right-continuous generalized inverse, and set* $J = \{s \geq 0, r_{t^-} < r_t\}$*. Then, for all* $t \in \mathbb{R}^+\backslash J$*,* $\lim\limits_{t\to+\infty} r^n_t = r_t$*.*

2. *If* $(a^n_t)_{t\geq 0}$ *converges (locally) uniformly to some strictly increasing function* $(a_t)_{t\geq 0}$ *such that* $\lim\limits_{t\to+\infty} a_t = +\infty$*, then* $(r^n_t)_{t\geq 0}$ *converges (locally) uniformly to* $(r_t)_{t\geq 0}$*, the inverse of* $(a_t)_{t\geq 0}$*.*

Lemma A.0.5. *Consider a Brownian motion* $(W_t)_{t\geq 0}$ *and denote by* $(L^x_t)_{t\geq 0}$ *its local time at* $x \in \mathbb{R}$*. Then for all* $T > 0$*,* $\sup\limits_{[0,T]\times\mathbb{R}} L^x_t$ *is almost surely finite.*

Proof. Fix $T > 0$ and $t \in [0,T]$ and $x \in \mathbb{R}$. Firstly, one has $L^x_t \leq L^x_T$. Moreover, by Tanaka formula,

$$L^x_T = |W_T - x| - |x| - \int_0^T \mathrm{sgn}(W_s - x)\,\mathrm{d}W_s \leq |W_T| + |x| - |x| + \int_0^T |\mathrm{d}W_s|\,.$$

Thus, $\sup\limits_{[0,T]\times\mathbb{R}} L^x_t \leq \sup\limits_{\mathbb{R}} L^x_T \leq 2\,|W_T| < +\infty$ almost surely. □

Appendix B

Scilab code

Here is the main code used to do the simulation.

```
function [MB]=Brownian_motion(t,N)
        h=t/N
        acc = grand(1,N,"nor",0,sqrt(h))
        MB=zeros(1,N+1)
        for k=2:N+1
                MB(k) = MB(k-1)+acc(k-1)
        end
endfunction
```

Listing B.1: To simulate a standard Brownian motion

```
function []=distribution_Xe(M,epsilon,t,N,rho,alpha,bet)
        // M is the number of simulations
        h=t/N
        a=(alpha+1)/2-bet
        Y=[]
        for i=1:M
                MB=Brownian_motion(t,N)
                G=grand(1,N+1,"nor",0,1)
                x=[0:h:t]
                y=zeros(G)
                for k=1:N
                        y(k+1)=(2*rho*(abs(G(k+1)))^(alpha))*sign(G(k+1))
                        *exp((a*k*h)/(epsilon))
                end
                j=inttrap(x,y)
                Yt=-2*sqrt(epsilon)*G(N+1)+2*MB(N+1)+j/sqrt(epsilon)
                Y=[Y,Yt]
        end
        histplot(100,Y)
        z=-16:0.1:16
        s=4*t
        plot2d(z,exp(-z.^2/(2*s))/sqrt(2*%pi*s),2)
endfunction
```

Listing B.2: To print the distribution of $\sqrt{\epsilon}X^{(e)}_{t/\epsilon}$.

Remark B.0.1. It uses (3.4), where $V^{(e)}_{t/\epsilon}$ is approximated by a normal distribution.

```
function []=distribution_X_with_OU(M,epsilon,t,N,rho,alpha,bet)
```

```
        // M is the number of simulations
        h=log(t/epsilon)/N
        a=(alpha+1)/2-bet
        speed=(epsilon)^(1.5)
        Y=[]
        for i=1:M
                x=[0:h:log(t/epsilon)]
                y=zeros(x)
                for k=1:N
                        OU=grand(1,1,"nor",0,sqrt(1-exp(-k*h)))
                        y(k+1)=OU*exp(3*k*h/2)
                end
                j=inttrap(x,y)
                Yt=speed*j
                Y=[Y,Yt]
        end
        z=-34:0.1:34
        histplot(100,Y)
        s=t^3/3
        plot2d(z,exp(-z.^2/(2*s))/sqrt(2*%pi*s),17)
endfunction
```

Listing B.3: To print the distribution of $\epsilon^{3/2} X_{t/\epsilon} \approx \epsilon^{3/2} \int_0^{\log(t/\epsilon)} \tilde{V}_s^{(e)} e^{3s/2} du$.

```
function []=distribution_X(M,epsilon,t,N,rho,alpha,bet)
        // M is the number of simulations
        h=log(t/epsilon)/N
        a=(alpha+1)/2-bet
        speed=(epsilon)^(1.5)
        Y=[]
        for l=1:M
                // Computation of \tilde(V):
                S1=[]
                for k=1:N
                        for j=1:N
                                S1=[S1,j*k*h/N]
                        end
                end
                S=unique(S1)
                Vtilde=[]
                for i=1:length(S)
                        v=grand(1,1,"nor",0,sqrt(1-exp(-S(i))))
                        Vtilde=[Vtilde,v]
                end
                x=[0:h:log(t/epsilon)]
                y=zeros(x)
                for k=1:N
                        s=k*h
                        //computation of U(s)
                        h2=s/N
                        x2=[0:h2:s]
                        u=zeros(x2)
                        for j=1:N
                                i=find(S==j*k*h/N)
```

```
                    v2=Vtilde(i)
                    u(j+1)=exp((a+0.5)*j*h2)*sign(v2)*abs(v2)^(alpha)
                end
                Us=exp(-s/2)*inttrap(x2,u)
                i=find(S==N*k*h/N)
                y(k+1)=(Vtilde(i)+Us)*exp(3*s/2)
            end
            I=inttrap(x,y)
            Yt=speed*I
            Y=[Y,Yt]
        end
        z=-34:0.1:34
        sig=t^3/3
        histplot(100,Y)
        plot2d(z,exp(-z.^2/(2*sig))/sqrt(2*%pi*sig),17);
endfunction
```

Listing B.4: To print the distribution of $\epsilon^{3/2}X_{t/\epsilon} \approx \epsilon^{3/2}\int_0^{\log(t/\epsilon)}(\tilde{V}_s^{(e)} + U_s)e^{3s/2}du$.

Remark B.0.2. The process $(U_t)_{t\geq 0}$ defined in Remark 3.2.2 has been approximated by

$$\tilde{U}_t := \int_0^t \rho e^{-(t-s)/2} e^{(\frac{\alpha+1}{2}-\beta)s}\,\mathrm{sgn}(\tilde{V}_s^{(e)})\left|\tilde{V}_s^{(e)}\right|^{\alpha} \mathrm{d}s.$$

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