On Wald-type optimal stopping for Brownian motion

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Optimal stopping problem

Let $G : \mathbb{R}^+ \to \mathbb{R}$ be a measurable map, satisfying

$$\forall x \in \mathbb{R}, \; G(|x|) \leq cx^2 + d, \tag{1}$$

for some $d \in \mathbb{R}$, $c > 0$.

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for some $d \in \mathbb{R}$, $c > 0$.
Goal: maximizing the expectation $\mathbb{E}\left[G(|B_\tau|) - c\tau\right]$, over all integrable $(\mathcal{F}_t)$-stopping times.

Wald's identity

For all integrable $(\mathcal{F}_t)$-stopping time τ,

$$\mathbb{E}\left[B_\tau^2\right] = \mathbb{E}[\tau].$$

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Proof: Martingale property for $(B_{t\wedge\tau}^2 - t\wedge\tau)_{t\geq 0}$:

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Doob's inequality: for all $t \geq 0$,

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So,

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Proposition

$$\sup_\tau \mathbb{E}\left[B_\tau^2 - c\tau\right] = \begin{cases} +\infty & \text{if } c \in]0,1[, \\ 0 & \text{elsewhere,} \end{cases}$$

where the supremum is taken over all integrable $(\mathcal{F}_t)$-stopping times.

Theorem

Let $0 < p < 2$ and $c > 0$, we have,

$$\sup_{\tau} \mathbb{E}\left[|B_\tau|^p - c\tau\right] = \frac{2-p}{p}\left(\frac{p}{2c}\right)^{p/(2-p)},$$

where the supremum is taken over all integrable $(\mathcal{F}_t)$-stopping times.
The optimal stopping time is
$\tau_{p,c} = \inf\left\{t \geq 0, |B_t| = \left(\frac{p}{2c}\right)^{1/(2-p)}\right\}.$

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Optimality in the first point: $T_{x_0} = \inf\{t \geq 0, |B_t| = x_0\}$

$$V_{T_{x_0}}(G, c) = \mathbb{E}\left[G\left(\left|B_{T_{x_0}}\right|\right) - cB^2_{T_{x_0}}\right]$$

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\begin{aligned}
V_{T_{x_0}}(G,c) &= \mathbb{E}\left[G\left(\left|B_{T_{x_0}}\right|\right) - cB^2_{T_{x_0}}\right] \\
&= D_{G,c}(-x_0)\mathbb{P}\left(B_{T_{x_0}} = -x_0\right) + D_{G,c}(x_0)\mathbb{P}\left(B_{T_{x_0}} = x_0\right)
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Theorem

The solution to our optimal stopping problem is

$$\sup_{\tau} \mathbb{E}\left[G(|B_\tau|) - c\tau\right] = \sup_{x\in\mathbb{R}} (G(|x|) - cx^2),$$

where the supremum is taken over all integrable $(\mathcal{F}_t)$*-stopping times.*
When $D_{G,c}$ *reaches a maximum on* $\mathbb{R}$*, the optimal stopping time* $\tau_{G,c} = \inf\{t \geq 0, |B_t| = \mathrm{argmax}\ D_{G,c}\}$.

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When $D_{G,c}$ reaches a maximum on $\mathbb{R}$, the optimal stopping time $\tau_{G,c} = \inf\{t \geq 0, |B_t| = \mathrm{argmax}\ D_{G,c}\}$.

Example: $G : x \mapsto \ln(x)$

$D : x \mapsto \ln(|x|) - cx^2$ reaches a maximum at $\pm\frac{1}{\sqrt{2c}}$.

$$\sup_{\tau} \mathbb{E}\left[\ln(|B_\tau|) - c\tau\right] = -\frac{1}{2}(\ln(2c) + 1).$$

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For all integrable $(\mathcal{F}_t)$-stopping time τ, if $p \in]0,2[$,

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Let $f : c \mapsto c\mathbb{E}[\tau] + \frac{2-p}{p}\left(\frac{p}{2c}\right)^{p/(2-p)}$. f reaches a minimum at $\frac{p}{2}\mathbb{E}[\tau]^{(p-2)/2}$ and $f\left(\frac{p}{2}\mathbb{E}[\tau]^{(p-2)/2}\right) = \mathbb{E}[\tau]^{p/2}$.

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$g : c \mapsto c\mathbb{E}[\tau] - \frac{1}{2}(\ln(2c) + 1)$ reaches a minimum at $c = \frac{1}{2\mathbb{E}[\tau]}$.
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Optimality

If that there exists $c_0 > 0$ such that $D_{G,c_0} : x \mapsto G(|x|) - c_0x^2$ reaches a maximum over $\mathbb{R}$, then

$$\sup_\tau\left(\mathbb{E}\left[G(|B_\tau|)\right] - \inf_{c>0}\left(c\mathbb{E}[\tau] + \sup_{x\in\mathbb{R}}(G(|x|) - cx^2)\right)\right) = 0,$$

where the supremum is taken over all integrable $(\mathcal{F}_t)$-stopping times.

Proof: Call $a_{G,c}$ a point where $D_{G,c}$ reaches its maximum (possibly infinite) over $\bar{\mathbb{R}}$, $\sigma_c = \inf\{t \geq 0,\ |B_t| = a_{G,c}\}$.

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$$0 = \mathbb{E}\left[G\left(\left|B_{\sigma_{c_0}}\right|\right) - c_0\sigma_{c_0}\right] - \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - c_0 x^2\right)$$

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&\leq \sup_{\tau}\sup_{c>0}\left(\mathbb{E}\left[G\left(|B_\tau|\right) - c\tau\right] - \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - cx^2\right)\right)
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&\leq \sup_{\tau}\sup_{c>0}\left(\mathbb{E}\left[G\left(|B_\tau|\right) - c\tau\right] - \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - cx^2\right)\right) \\
&\leq \sup_{\tau}\left(\mathbb{E}\left[G\left(|B_\tau|\right)\right] + \sup_{c>0}\left(-c\mathbb{E}[\tau] - \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - cx^2\right)\right)\right)
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&\leq \sup_{c>0}\left(\mathbb{E}\left[G\left(\left|B_{\sigma_{c_0}}\right|\right) - c\sigma_{c_0}\right] - \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - cx^2\right)\right) \\
&\leq \sup_{\tau}\sup_{c>0}\left(\mathbb{E}\left[G\left(|B_\tau|\right) - c\tau\right] - \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - cx^2\right)\right) \\
&\leq \sup_{\tau}\left(\mathbb{E}\left[G\left(|B_\tau|\right)\right] + \sup_{c>0}\left(-c\mathbb{E}[\tau] - \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - cx^2\right)\right)\right) \\
&\leq \sup_{\tau}\left(\mathbb{E}\left[G\left(|B_\tau|\right)\right] - \inf_{c>0}\left(c\mathbb{E}[\tau] + \sup_{x\in\mathbb{R}}\left(G\left(|x|\right) - cx^2\right)\right)\right).
\end{aligned}$$

Proposition

If τ is an integrable $(\mathcal{F}_t)$-stopping time, then

$$\mathbb{E}\left[\max_{0\leq t\leq\tau} B_t\right] \leq \sqrt{\mathbb{E}[\tau]}.$$

The equality is reached for $\tau = \inf\{t \geq 0, S_t - B_t = a\}$, *for* $a > 0$.

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$(S_t - B_t)$ has the same law as $|B_t| \rightsquigarrow (Z_t)_{t\geq 0}$ is a martingale.

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Let σ be a bounded $(\mathcal{F}_t)$-stopping time, $\mathbb{E}[B_\sigma] = 0$,

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So $\mathbb{E}[S_\sigma] \leq \inf_{c>0}\left(\frac{1}{4c} + c\mathbb{E}[\sigma]\right)$.

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$$\text{Hence, } \forall t \geq 0,\ \mathbb{E}[S_{t\wedge\tau}] \leq \sqrt{\mathbb{E}[t \wedge \tau]}.$$

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Conclude with monotone convergence theorem.

Sharpness:

Let $a \in \mathbb{R}$, take $\tau = \inf\{t \geq 0, S_t - B_t = a\}$ which is equal in law to $T_a = \inf\{t \geq 0, |B_t| = a\}$.

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$$\mathbb{E}[\tau] = \mathbb{E}[T_a] = a^2.$$

Proposition

If τ is an integrable $(\mathcal{F}_t)$-stopping time, then

$$\mathbb{E}\left[\max_{0\leq t\leq \tau} |B_t|\right] \leq \sqrt{2}\sqrt{\mathbb{E}[\tau]}.$$

The equality is reached for
$\tau_2 = \inf\{t \geq 0, \max_{0\leq s\leq t} |B_s| - |B_t| = a\}$, *for* $a > 0$.

Thank you !

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